

LINEAR PROGRAMMING APPROACH TO DETERMINISTIC LONG RUN AVERAGE PROBLEMS OF OPTIMAL CONTROL*

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Abstract. We establish that deterministic long run average problems of optimal control are “asymptotically equivalent” to infinite-dimensional linear programming problems (LPPs) and we establish that these LPPs can be approximated by finite-dimensional LPPs, the solutions of which can be used for construction of the optimal controls. General results are illustrated with numerical examples.

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1. Introduction and description of the problems. In this paper we show that, under some conditions, deterministic long run average problems of optimal control are “asymptotically equivalent” to infinite-dimensional linear programming problems (LPPs) and we establish that these LPPs can be approximated by finite-dimensional LPPs, the solutions of which can be used for numerical construction of the optimal controls.

Infinite horizon problems of optimal control have been studied intensively in both deterministic and stochastic settings (see Anderson and Kokotovic [3], Arisawa, Ishii, and Lions [5], Bardi and Capuzzo-Dolcetta [10], Bensoussan [12], Carlson, Haurie, and Leizarowitz [14], Colonius and Kliemann [17], Fleming and Soner [21], Grüne [29], Kushner [34], Kushner and Dupuis [35], Vigodner [46], and references therein). In the stochastic setting, the linear programming formulation is a common tool for treating the problems (see, e.g., Basak, Borkar, and Ghosh [11], Borkar [13], Hernandez-Lerma and Lasserre [31], Stockbridge [44], Yin and Zhang [48]). Finite-dimensional approximations of LPPs arising in stochastic optimal control problems were considered by Helmes and Stockbridge [30] and by Mendiondo and Stockbridge [38]. A linear programming approach to long run average optimal control problems in the deterministic setting appears to be new and, to the best of our knowledge, there are no publications devoted to this topic (under different assumptions and for a different problem, a linear programming formulation was discussed in Evans and Gomes [20]). A linear programming approach to deterministic optimal control problems on a finite time interval has been studied in Rubio [42].

Let us introduce the problems that we will be dealing with. Consider the control system

$$(1) \quad \dot{y}(\tau) = f(u(\tau), y(\tau)), \quad \tau \in [0, S],$$

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where the function $f(u, y) : U \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is continuous in (u, y) and satisfies Lipschitz conditions in y ; the controls are Lebesgue measurable functions $u(\tau) : [0, S] \rightarrow U$ and U is a compact metric space.

A pair $(u(\tau), y(\tau))$ is called *admissible* on the interval $[0, S]$ if (1) is satisfied for almost all $\tau \in [0, S]$ and $y(\tau) \in Y \ \forall \tau \in [0, S]$, where Y is a given compact subset of \mathbb{R}^m . The pair is called admissible on $[0, \infty)$ if it is admissible on any interval $[0, S]$, $S > 0$.

Let $g(u, y) : U \times \mathbb{R}^m \rightarrow \mathbb{R}^1$ be a continuous function. We will be considering the asymptotics of the optimal control problem

$$(2) \quad \frac{1}{S} \inf_{(u(\cdot), y(\cdot))} \int_0^S g(u(\tau), y(\tau)) d\tau \stackrel{\text{def}}{=} G(S),$$

where inf is over all admissible pairs on the interval $[0, S]$. Along with (2), we will be referring to the infinite time horizon optimal control problem

$$(3) \quad \inf_{(u(\cdot), y(\cdot))} \lim_{S \rightarrow \infty} \frac{1}{S} \int_0^S g(u(\tau), y(\tau)) d\tau \stackrel{\text{def}}{=} G_\infty,$$

where inf is over all admissible pairs on the interval $[0, \infty)$ such that the limit in the above expression exists. If this inf is sought over the periodic admissible pairs only, that is, over the pairs such that

$$(4) \quad (u(\tau), y(\tau)) = (u(\tau + T), y(\tau + T)) \ \forall \tau \geq 0$$

for some $T > 0$, then (3) becomes equivalent to a so-called periodic optimization problem (see, e.g., Colonius [15])

$$(5) \quad \inf_{(u(\cdot), y(\cdot))} \frac{1}{T} \int_0^T g(u(\tau), y(\tau)) d\tau \stackrel{\text{def}}{=} G_{per},$$

where inf is over the length of the time interval T and over the admissible pairs defined on $[0, T]$ which satisfy the periodicity condition $y(0) = y(T)$.

A very special family of admissible pairs on $[0, \infty)$ is that consisting of constant valued controls and corresponding steady state solutions of (1):

$$(6) \quad (u(\tau), y(\tau)) = (u, y) \in M \stackrel{\text{def}}{=} \{(u, y) \mid (u, y) \in U \times Y, f(u, y) = 0\}.$$

If inf is sought over the admissible pairs from this family, the problem (3) is reduced to

$$(7) \quad \inf_{(u, y) \in M} g(u, y) \stackrel{\text{def}}{=} G_{ss},$$

which is called a steady state optimization problem. It is easy to see that the optimal values of the above introduced problems satisfy the inequalities

$$(8) \quad \overline{\lim}_{S \rightarrow \infty} G(S) \leq G_\infty \leq G_{per} \leq G_{ss}.$$

The approach that we are developing in the paper is based on a reformulation of problem (2) as the problem of minimization over the set of occupational measures generated on the interval $[0, S]$ by the admissible pairs of (1) and on the fact that this set is proven to converge (as $S \rightarrow \infty$) to a set of probability measures characterized by

linear constraints (as has been recently established in [26]). Note that it is the presence of this convergence that constitutes the main difference between our approach and a linear programming approach to deterministic optimal control problems on a finite time interval developed by Rubio [42].

Note also in conclusion that results obtained in the paper have a potential for applications in asymptotic and numerical analysis of singularly perturbed control systems (**SPCS**), which have been the focus of many researchers (see Alvarez and Bardi [1, 2], Artstein [6, 7], Colonius and Fabbri [16], Donchev and Dontchev [19], Gaitsgory [26], Grammel [28], Kabanov and Pergamenschikov [32], Leizarowitz [36], Naidu [39], and Quincampoix and Watbled [41] for the most recent developments and also for references to earlier results in the area). One such application follows directly from the fact that tending S to infinity in problem (2) is equivalent to tending ϵ to zero in the problem

$$(9) \quad \inf_{(u^\epsilon(\cdot), y^\epsilon(\cdot))} \int_0^1 g(u^\epsilon(t), y^\epsilon(t)) dt \stackrel{\text{def}}{=} G(\epsilon)$$

considered on the admissible pairs $(u^\epsilon(\cdot), y^\epsilon(\cdot)) \in U \times Y$ of the SPCS

$$(10) \quad \epsilon \frac{dy^\epsilon(t)}{dt} = f(u^\epsilon(t), y^\epsilon(t)),$$

where (9) and (10) are obtained from (2) and (1) with $\epsilon \stackrel{\text{def}}{=} \frac{1}{S}$ and with $t = \tau\epsilon$, $u^\epsilon(t) = u(\frac{t}{\epsilon})$, $y^\epsilon(t) = y(\frac{t}{\epsilon})$. By formally taking $\epsilon = 0$ in (9)–(10), one obtains the so-called reduced problem, which proves to be equivalent to the steady state optimization problem (7). This implies that the statement about the validity of the equality $\lim_{\epsilon \rightarrow 0} G(\epsilon) = G(0)$, which can be interpreted as a weak version of Tichonov's theorem for the SPCS under consideration, is true only if the “less than or equal to” inequalities in (8) are satisfied as exact equalities. More elaborate connections between SPCS and long run average problems of optimal control implying the applicability of results of this paper in dealing with SPCS have been established in Alvarez and Bardi [1, 2], Artstein and Gaitsgory [8], and Gaitsgory [24, 25]; different, Tichonov-theorem-type results can be found in Kokotovic, Khalil, and O'Reilly [33], O'Malley [40], and Veliov [45].

The paper is organized as follows. In section 2, we give the occupational measures formulation of problem (2). In section 3, we show that, as S tends to infinity, the set of occupational measures converges to the set of probability measures with linear constraints, and we introduce the infinite-dimensional LPP defined on this set, which determines the asymptotics of problem (2) (Propositions 2 and 5, Corollaries 3 and 6). In section 4, we establish that the infinite-dimensional LPP can be approximated by a finite-dimensional LPP (Propositions 7 and 9). In section 5, we discuss the possibility of using the solution of the latter to construct an approximation to the solution of the periodic optimization problem (5) and we also illustrate the idea of the construction with two numerical examples. The proofs for sections 3, 4, and 5 are given in section 6.

2. Occupational measures formulation. Let $\mathcal{P}(U \times Y)$ stand for the space of probability measures defined on the Borel subsets of $U \times Y$. Given an arbitrary admissible (on the interval $[0, S]$) pair $(u(\tau), y(\tau))$, one can define a probability measure $\gamma^{(u(\cdot), y(\cdot))} \in \mathcal{P}(U \times Y)$ by taking

$$(11) \quad \gamma^{(u(\cdot), y(\cdot))}(Q) \stackrel{\text{def}}{=} \frac{1}{S} \text{meas} \left\{ \tau \mid (u(\tau), y(\tau)) \in Q \right\}$$

for any Borel $Q \subset U \times Y$, where $\text{meas}\{\cdot\}$ stands for the Lebesgue measure on $[0, S]$. Such a probability measure is called the *occupational measure* generated by the pair $(u(\tau), y(\tau))$. Note that the occupational measure generated by a steady state admissible pair $(u(\tau), y(\tau)) = (u, y) \in M$ (as in (6)) is just the Dirac measure at (u, y) .

It is easy to see that (11) is equivalent to the equality

$$(12) \quad \int_{U \times Y} \chi_Q(u, y) \gamma^{(u(\cdot), y(\cdot))}(du, dy) = \frac{1}{S} \int_0^S \chi_Q(u(\tau), y(\tau)) d\tau,$$

where $\chi_Q(\cdot)$ is the indicator function of the set Q : $\chi_Q(u, y) = 1 \quad \forall (u, y) \in Q$ and $\chi_Q(u, y) = 0 \quad \forall (u, y) \notin Q$. The validity of (12) for any indicator function leads to the validity of a similar equality for the simple functions (that is, linear combinations of the indicator functions) and, thus, with the help of a standard approximation argument, leads to the validity of the equality

$$(13) \quad \int_{U \times Y} q(u, y) \gamma^{(u(\cdot), y(\cdot))}(du, dy) = \frac{1}{S} \int_0^S q(u(\tau), y(\tau)) d\tau$$

for any continuous function $q(u, y) : U \times \mathbb{R}^m \rightarrow \mathbb{R}^1$.

Denote by $\Gamma(S) \subset \mathcal{P}(U \times Y)$ the set of all occupational measures generated by the admissible pairs on the interval $[0, S]$. Using this notation and (13), one can rewrite problem (2) in the equivalent form

$$(14) \quad \inf_{\gamma \in \Gamma(S)} \int_{U \times Y} g(u, y) \gamma(du, dy) = G(S).$$

In what follows, the convergence properties of $G(S)$ (as S tends to infinity) are established on the basis of the corresponding convergence properties of $\Gamma(S)$. To describe these convergence properties, let us introduce a metric ρ on $\mathcal{P}(U \times Y)$ as follows:

$$(15) \quad \rho(\gamma', \gamma'') \stackrel{\text{def}}{=} \sum_{j=1}^{\infty} \frac{1}{2^j} \left| \int_{U \times Y} q_j(u, y) \gamma'(du, dy) - \int_{U \times Y} q_j(u, y) \gamma''(du, dy) \right|$$

$\forall \gamma', \gamma'' \in \mathcal{P}(U \times Y)$, where $q_j(\cdot), j = 1, 2, \dots$, is a sequence of Lipschitz continuous functions which is dense in the unit ball of $C(U \times Y)$ (the space of continuous functions on $U \times Y$). Note that this metric is consistent with the weak convergence topology of $\mathcal{P}(U \times Y)$. Namely, a sequence $\gamma^k \in \mathcal{P}(U \times Y)$ converges to $\gamma \in \mathcal{P}(U \times Y)$ in this metric if and only if

$$(16) \quad \lim_{k \rightarrow \infty} \int_{U \times Y} q(u, y) \gamma^k(du, dy) = \int_{U \times Y} q(u, y) \gamma(du, dy)$$

for any continuous $q(u, y) : U \times Y \rightarrow \mathbb{R}^1$. Note also that, the space $\mathcal{P}(U \times Y)$ is known to be compact in its weak convergence topology and, hence, being equipped with the metric (15), it becomes a compact metric space.

Using the metric ρ , one can define the “distance” $\rho(\gamma, \Gamma)$ between $\gamma \in \mathcal{P}(U \times Y)$ and $\Gamma \subset \mathcal{P}(U \times Y)$ and define the Hausdorff metric $\rho_H(\Gamma_1, \Gamma_2)$ between $\Gamma_1 \subset \mathcal{P}(U \times Y)$ and $\Gamma_2 \subset \mathcal{P}(U \times Y)$ as follows:

$$(17) \quad \rho(\gamma, \Gamma) \stackrel{\text{def}}{=} \inf_{\gamma' \in \Gamma} \rho(\gamma, \gamma'), \quad \rho_H(\Gamma_1, \Gamma_2) \stackrel{\text{def}}{=} \max \left\{ \sup_{\gamma \in \Gamma_1} \rho(\gamma, \Gamma_2), \sup_{\gamma \in \Gamma_2} \rho(\gamma, \Gamma_1) \right\}.$$

The following simple lemma is implied by the definitions above.

LEMMA 1. *Let Γ be a subset of $\mathcal{P}(U \times Y)$.*

(i) *If $\lim_{S \rightarrow \infty} \sup_{\gamma \in \Gamma(S)} \rho(\gamma, \Gamma) = 0$, then, for any continuous $q(u, y) : U \times Y \rightarrow \mathbb{R}^1$,*

$$\underline{\lim}_{S \rightarrow \infty} \inf_{\gamma \in \Gamma(S)} \int_{U \times Y} q(u, y) \gamma(du, dy) \geq \inf_{\gamma \in \Gamma} \int_{U \times Y} q(u, y) \gamma(du, dy).$$

(ii) *If $\lim_{S \rightarrow \infty} \rho_H(\Gamma(S), \Gamma) = 0$, then*

$$\lim_{S \rightarrow \infty} \inf_{\gamma \in \Gamma(S)} \int_{U \times Y} q(u, y) \gamma(du, dy) = \inf_{\gamma \in \Gamma} \int_{U \times Y} q(u, y) \gamma(du, dy).$$

Proof. The proof is obvious. \square

3. Infinite-dimensional LPPs. Define the set $W \subset \mathcal{P}(U \times Y)$ by the equation

$$(18) \quad W \stackrel{\text{def}}{=} \left\{ \gamma : \gamma \in \mathcal{P}(U \times Y); \int_{U \times Y} (\phi'(y))^T f(u, y) \gamma(du, dy) = 0 \quad \forall \phi(\cdot) \in C^1 \right\},$$

where C^1 is the space of continuously differentiable functions $\phi(y) : \mathbb{R}^m \rightarrow \mathbb{R}^1$ and $\phi'(y)$ is a vector column of partial derivatives (the gradient) of $\phi(y)$.

Note that the set W can be empty. It is easy to see, for example, that W is empty if there exists a continuously differentiable function $\phi(\cdot) \in C^1$ such that

$$(19) \quad \max_{(u, y) \in U \times Y} (\phi'(y))^T f(u, y) < 0.$$

The set W is not empty if the set of steady state or periodic admissible pairs is not empty since the occupational measure generated by each such pair is contained in W . In fact, let $(u(\cdot), y(\cdot))$ be a periodic admissible pair (that is, (4) is satisfied with some positive T) and let $\gamma^{(u(\cdot), y(\cdot))}$ be the occupational measure generated by this pair on the interval $[0, T]$. Then, by (13),

$$\begin{aligned} \int_{U \times Y} (\phi'(y))^T f(u, y) \gamma^{(u(\cdot), y(\cdot))}(du, dy) &= \frac{1}{T} \int_0^T (\phi'(y(\tau)))^T f(u(\tau), y(\tau)) d\tau \\ &= \frac{\phi(y(T)) - \phi(y(0))}{T} = 0 \quad \forall \phi(\cdot) \in C^1 \quad \Rightarrow \quad \gamma^{(u(\cdot), y(\cdot))} \in W. \end{aligned}$$

PROPOSITION 2. *If the set W is empty, then there exists $S_0 > 0$ such that $\Gamma(S)$ is empty for $S \geq S_0$. If $\Gamma(S)$ is not empty for $S > 0$, then W is not empty and*

$$(20) \quad \lim_{S \rightarrow \infty} \sup_{\gamma \in \Gamma(S)} \rho(\gamma, W) = 0.$$

Proof. The proof is similar to the corresponding part of the proof of Theorem 2.1(i) in [26]. For the sake of completeness, we have displayed it in section 6. \square

Assume that W is not empty and consider the problem

$$(21) \quad \min_{\gamma \in W} \int_{U \times Y} g(u, y) \gamma(du, dy) \stackrel{\text{def}}{=} G^*,$$

where $g(\cdot)$ is the same as in (14) (and the same as in (2)–(7)). It can be easily seen that the set W is convex and compact. Moreover, since both the objective function and the constraints defining W are linear in γ , problem (21) is that of infinite-dimensional linear programming (see, e.g., [4]).

COROLLARY 3. *The lower limit of the optimal values of (2) satisfies the inequality*

$$\liminf_{S \rightarrow \infty} G(S) = \liminf_{S \rightarrow \infty} \inf_{\gamma \in \Gamma(S)} \int_{U \times Y} g(u, y) \gamma(du, dy) \geq G^*.$$

Proof. The proof follows from Lemma 1(i), Proposition 2, and the validity of the representation (14). \square

COROLLARY 4 (criteria of optimality). (i) *If an admissible pair $(u(\cdot), y(\cdot)) : [0, \infty) \rightarrow U \times Y$ is such that*

$$\lim_{S \rightarrow \infty} \frac{1}{S} \int_0^S g(u(\tau), y(\tau)) d\tau = G^*,$$

then this pair is a solution of problem (3) and $G_\infty = G^$.*

(ii) *If a periodic (with a period T) admissible pair $(u(\cdot), y(\cdot))$ is such that*

$$\frac{1}{T} \int_0^T g(u(\tau), y(\tau)) d\tau = G^*,$$

then this pair is a solution of problems (3) and (5), and also $G_\infty = G_{per} = G^$.*

(iii) *If a steady state admissible pair $(u(\tau), y(\tau)) = (u, y) \in M$ (as defined in (6)) is such that*

$$g(u, y) = G^*,$$

then this pair is a solution of problems (3), (5), and (7), and also $G_\infty = G_{per} = G_{ss} = G^$.*

Proof. The proof follows from inequalities (8) and Corollary 3. \square

Denote by $\mathcal{P}(U)$ the space of probability measures defined on the Borel subsets of U and consider the system

$$(22) \quad \dot{y}(\tau) = \bar{f}(\nu(\tau), y(\tau)), \quad \tau \in [0, S],$$

where $\nu(\tau) \in \mathcal{P}(U)$ are relaxed controls (see [47]) and $\bar{f}(\nu, u) \stackrel{\text{def}}{=} \int_U f(u, y) \nu(du)$.

A pair $(\nu(\tau), y(\tau))$ will be called *relaxed admissible* on the interval $[0, S]$ if (22) is satisfied for almost all $\tau \in [0, S]$ and $y(\tau) \in Y \quad \forall \tau \in [0, S]$.

Assumption 1. For any Lipschitz continuous function $q(u, y) : U \times \mathbb{R}^m \rightarrow \mathbb{R}^1$,

$$(23) \quad \left| \frac{1}{S} \sup_{(u(\cdot), y(\cdot))} \int_0^S q(u(\tau), y(\tau)) d\tau - \frac{1}{S} \sup_{(\nu(\cdot), y(\cdot))} \int_0^S \bar{q}(\nu(\tau), y(\tau)) d\tau \right| \stackrel{\text{def}}{=} \alpha_q(S) \rightarrow 0$$

as $S \rightarrow \infty$, where $\bar{q}(\nu, u) \stackrel{\text{def}}{=} \int_U q(u, y) \nu(du)$, with the first sup being over all admissible pairs and the second being over all relaxed admissible pairs.

Remark 1. The fulfillment of Assumption 1 is related to the applicability of Filippov–Wazewski type theorems on Y (see Frankowska and Rampazo [22]). In particular, it is satisfied with $\alpha_q(S) \equiv 0$ if Y is forward invariant with respect to the

solutions of system (1), that is, if for an arbitrary control $u(\tau)$, any solution $y(\tau)$ of (1) with the initial conditions in Y does not leave Y (see, e.g., Theorem 10.4.4 in Aubin and Frankowska [9]). Assumption 1 is also satisfied with $\alpha_q(S) \equiv 0$ if $f(u, y) \equiv f(y)$ (the case of uncontrolled dynamics, with inf's in (2)–(5) being over the admissible trajectories having different initial conditions). In this case U can be formally defined to consist of only one point and systems (1) and (22) are identical.

Assumption 1 is not satisfied if, for example, the set of admissible pairs is empty, while the set of relaxed admissible pairs is not, as in the case when $m = 1$, $f(u, y) = -y + u$, with U consisting of two points $U = \{-1, 1\}$, and Y consisting of one point $Y = \{0\}$.

PROPOSITION 5. *Let $\Gamma(S)$ be nonempty and Assumption 1 be satisfied. Then*

$$(24) \quad \lim_{S \rightarrow \infty} \rho_H(\text{co}\Gamma(S), W) = 0,$$

where $\text{co}\Gamma(S)$ stands for the convex hull of $\Gamma(S)$.

Proof. The proof of (24) is similar to the proof of Theorem 1(i) in [26], which was established under a stronger assumption implying the validity of Assumption 1. The necessary adjustments for the case under consideration are made in section 6. \square

COROLLARY 6. *If Assumption 1 is satisfied, then the limit of the optimal value of (2) exists and is equal to G^* ,*

$$(25) \quad \lim_{S \rightarrow \infty} G(S) = G^*.$$

Also, if the solution γ^ of problem (21) is unique, then, for any $\gamma^S \in \Gamma(S)$ such that $\lim_{S \rightarrow \infty} \int_{U \times Y} g(u, y) \gamma^S(du, dy) = G^*$,*

$$(26) \quad \lim_{S \rightarrow \infty} \rho(\gamma^S, \gamma^*) = 0.$$

Proof. Since

$$\inf_{\gamma \in \text{co}\Gamma(S)} \int_{U \times Y} g(u, y) \gamma(du, dy) = \inf_{\gamma \in \Gamma(S)} \int_{U \times Y} g(u, y) \gamma(du, dy),$$

then, by (14),

$$\inf_{\gamma \in \text{co}\Gamma(S)} \int_{U \times Y} g(u, y) \gamma(du, dy) = G(S).$$

The validity of (25) follows now from Lemma 1(ii) and Proposition 5. The validity of (26) is, in turn, implied by (25) and Proposition 2. \square

Note that the solution γ^* of problem (21) can be unique only if it is an extreme point of W (since (21) is an LPP) and that, using (24), one can show (although not shown here) that, for any extreme point γ of W , there exists $\gamma^S \in \Gamma(S)$ such that $\lim_{S \rightarrow \infty} \rho(\gamma^S, \gamma) = 0$.

Let γ^* be a solution of problem (21) which is an extreme point of W and let $\gamma^S \in \Gamma(S)$ satisfy (26). Assume that there exists an admissible pair $(u^{\gamma^*}(\cdot), y^{\gamma^*}(\cdot)) : [0, \infty) \rightarrow U \times Y$ that generates γ^* on any interval $[0, S]$ (we will say that γ^* is generated by the pair on $[0, \infty)$ in this case). Then, for any continuous $q(u, y) : U \times Y \rightarrow \mathbb{R}^1$,

$$(27) \quad \lim_{S \rightarrow \infty} \frac{1}{S} \int_0^S q(u^{\gamma^*}(\tau), y^{\gamma^*}(\tau)) d\tau = \int_{U \times Y} q(u, y) \gamma^*(du, dy)$$

and, in particular, for $q(u, y) = g(u, y)$,

$$\lim_{S \rightarrow \infty} \frac{1}{S} \int_0^S g(u^{\gamma^*}(\tau), y^{\gamma^*}(\tau)) d\tau = \int_{U \times Y} g(u, y) \gamma^*(du, dy) = G^*.$$

Thus, by Corollary 4(i), this pair will be a solution of problem (3). Also, by Corollary 4(ii), (iii), this pair will be a solution of the periodic optimization problem (5) (and the steady state problem (7)) if it proves to be periodic (and, respectively, steady state).

4. Finite-dimensional approximations. Let $\{\phi_i(\cdot), i = 1, 2, \dots\}$ be a sequence of continuously differentiable functions such that any function $\phi(\cdot) \in C^1$ and its gradient $\phi'(\cdot)$ can be simultaneously approximated on Y by linear combinations of functions from $\{\phi_i(\cdot), i = 1, 2, \dots\}$ and their corresponding gradients. That is, for any $\phi(\cdot) \in C^1$ and any $\delta > 0$, there exist β_1, \dots, β_k (real numbers) such that

$$\max_{y \in Y} \left\{ \left| \phi(y) - \sum_1^k \beta_i \phi_i(y) \right| + \left\| \phi'(y) - \sum_1^k \beta_i \phi'_i(y) \right\| \right\} \leq \delta,$$

with $\|\cdot\|$ being a norm in \mathbb{R}^m . An example of such an approximating sequence is the sequence of monomials $y_1^{i_1} \dots y_m^{i_m}, i_1, \dots, i_m = 0, 1, \dots$, where $y_j (j = 1, \dots, m)$ stands for the j th component of y (see, e.g., [37]).

Using the system $\{\phi_i(\cdot), i = 1, 2, \dots\}$, one can represent the set W in the form of a countable system of equations:

$$(28) \quad W = \left\{ \gamma \mid \gamma \in \mathcal{P}(U \times Y); \int_{U \times Y} (\phi'_i(y))^T f(u, y) \gamma(du, dy) = 0, i = 1, 2, \dots \right\}.$$

Let us assume that the gradients $\phi'_i(\cdot), i = 1, \dots, N$, are linearly independent on any open ball B in \mathbb{R}^m (that is, the equality $\sum_{i=1}^N v_i \phi'_i(y) = 0 \forall y \in B$ can be valid only with $v_i = 0, i = 1, \dots, N$) and let us define the set W_N by truncation of the system of equations in (28):

$$(29) \quad W_N \stackrel{\text{def}}{=} \left\{ \gamma \mid \gamma \in \mathcal{P}(U \times Y); \int_{U \times Y} (\phi'_i(y))^T f(u, y) \gamma(du, dy) = 0, i = 1, 2, \dots, N \right\}.$$

Consider the LPP

$$(30) \quad \min_{\gamma \in W_N} \int_{U \times Y} q(u, y) \gamma(du, dy) \stackrel{\text{def}}{=} G_N.$$

Note that W_N is a convex and compact subset of $\mathcal{P}(U \times Y)$ and that $W \subset W_N$, which implies

$$(31) \quad G^* \geq G_N.$$

Note also that the set W_N is empty if (19) is true with $\phi(y) \stackrel{\text{def}}{=} \sum_{i=1}^N v_i \phi_i(y)$, where v_i are real numbers.

PROPOSITION 7. *The set W is not empty if and only if there exists $N_0 \geq 1$ such that W_N is not empty for $N \geq N_0$. If W is not empty, then*

$$(32) \quad \lim_{N \rightarrow \infty} \rho_H(W_N, W) = 0$$

and

$$(33) \quad \lim_{N \rightarrow \infty} G_N = G^*.$$

Also, if γ_N is a solution of problem (30) and $\lim_{N' \rightarrow \infty} \rho(\gamma_{N'}, \gamma) = 0$ for some subsequence of integers N' tending to infinity, then γ is a solution of (21). If the solution γ^* of problem (21) is unique, then, for any solution γ_N of (30), $\lim_{N \rightarrow \infty} \rho(\gamma_N, \gamma^*) = 0$.

Proof. By Lemma 1(ii), the validity of (33) follows from the validity of (32). The other statements included in the proposition readily follow from (32) and (33). The validity of (32) is established in section 6. \square

Let us introduce another assumption which we need to consider.

Assumption 2. The inequality

$$(34) \quad \sum_{i=1}^N v_i (\phi'_i(y))^T f(u, y) \leq 0 \quad \forall (u, y) \in U \times Y$$

is valid only with $v_i = 0 \quad \forall i = 1, \dots, N$.

This assumption is satisfied if there exists a closed subset Y^* of Y with a nonempty interior such that from the validity of (34) it follows that $\sum_{i=1}^N v_i \phi'_i(y) = 0 \quad \forall y \in Y^*$ (the equality of v_i to zero is implied in this case by linear independence of $\phi'_i(\cdot)$). The existence of such Y^* can be guaranteed, for instance, in two cases specified in the statement below.

PROPOSITION 8. *A closed set $Y^* \subset Y$ with a nonempty interior such that from the fact that*

$$(35) \quad (\phi'(y))^T f(u, y) \leq 0 \quad \forall (u, y) \in U \times Y$$

it follows that $\phi'(y) = 0 \quad \forall y \in Y^$ exists if one of the following conditions is satisfied:*

(i) *The set $f(y, U) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^m : x = f(y, u), u \in U\}$ is convex for $y \in Y$, and there exists $\bar{y} \in \text{int } Y$ such that*

$$(36) \quad 0 \in \text{int } f(\bar{y}, U),$$

where “int” stands for the interior of the corresponding set.

(ii) *There exists $Y^0 \subset Y$ such that the closure of Y^0 has a nonempty interior and such that any two points in Y^0 are connected by an admissible trajectory. That is, for any $y', y'' \in Y^0$, there exists an admissible pair $(u(\tau), y(\tau))$ defined on some interval $[0, S]$ such that $y(0) = y'$ and $y(S) = y''$.*

Proof. The proof is in section 6. \square

Remark 2. Note that Y^0 in Proposition 8(ii) can be equal to Y in which case Y is a subset of complete controllability of system (1) (see [29]). Note also that both Assumptions 1 and 2 can be easily verified if there exist positive definite matrices A_1 and A_2 such that

$$(37) \quad \begin{aligned} & (f(u, y') - f(u, y''))^T A_1 (y' - y'') \\ & \leq -(y' - y'')^T A_2 (y' - y'') \quad \forall y', y'' \in \mathbb{R}^m, \forall u \in U. \end{aligned}$$

The latter is a Liapunov-type stability condition that implies the validity of Assumption 3.1 in [24] and, thus, guarantees the existence of a compact set $Y^* \subset \mathbb{R}^m$, which is forward invariant with respect to the solutions of system (1) and which is the global attractor for the solutions of this system starting outside Y^* (Theorem 3.1(ii))

in [24]). The existence of such Y^* leads to the fulfillment of Assumption 1 in case $Y^* \subset Y$. Also, the set Y^* contains all periodic and steady state solutions of the system (Lemma 3.1 in [24]) and, moreover, it can be shown that Y^* is equal to the closure of the set of all points belonging to the periodic orbits. The latter implies the validity of Proposition 8(ii) (and, hence, the validity of Assumption 2) if the interior of Y^* is not empty. Condition (37) is satisfied, for example, if system (1) is linear (that is, $f(u, y) = Ay + Du$, with $u \in U \subset \mathbb{R}^n$ and A, D being matrices of the corresponding dimensions) and if the eigenvalues of A have negative real parts. The nonemptiness of the interior of Y^* can be guaranteed in this case if U has a nonempty interior and if the Kalman controllability matrix $\{D, AD, \dots, A^{m-1}D\}$ has the rank m .

Assume that, for any $\Delta > 0$, Borel sets $Q_{l,k}^\Delta \subset U \times Y$ ($l = 1, \dots, L^\Delta, k = 1, \dots, K^\Delta$) (called cells in what follows) are defined in such a way that two different cells do not intersect, the union of all cells is equal to $U \times Y$ and

$$(38) \quad \sup_{(u,y) \in Q_{l,k}^\Delta} \|(u, y) - (u_l, y_k)\| \leq c\Delta, \quad c = \text{const},$$

for some point $(u_l, y_k) \in Q_{l,k}^\Delta$, where, for simplicity of notation, it is assumed (from now on) that U is a compact subset of \mathbb{R}^n and $\|\cdot\|$ stands for a norm in \mathbb{R}^{n+m} . Fix these points (u_l, y_k) ($l = 1, \dots, L^\Delta, k = 1, \dots, K^\Delta$) and define a polyhedral set $W_N^\Delta \subset \mathbb{R}^{L^\Delta + K^\Delta}$ by the equation

$$(39) \quad W_N^\Delta \stackrel{\text{def}}{=} \left\{ \gamma = \{\gamma_{l,k}\} \geq 0 : \sum_{l,k} \gamma_{l,k} = 1, \right. \\ \left. \sum_{l,k} (\phi'_i(y_k))^T f(u_l, y_k) \gamma_{l,k} = 0, \quad i = 1, 2, \dots, N \right\},$$

where $\sum_{l,k} \stackrel{\text{def}}{=} \sum_{l=1}^{L^\Delta} \sum_{k=1}^{K^\Delta}$. Consider a finite-dimensional LPP

$$(40) \quad \min_{\gamma \in W_N^\Delta} \sum_{l,k} \gamma_{l,k} g(u_l, y_k) \stackrel{\text{def}}{=} G_N^\Delta.$$

Note that the set W_N^Δ is the set of probability measures on $U \times Y$ which assign nonzero probabilities only to the points (u_l, y_k) , and, as such,

$$(41) \quad W_N^\Delta \subset W_N \quad \Rightarrow \quad G_N^\Delta \geq G_N.$$

PROPOSITION 9. *Let Assumption 2 be satisfied. Then the set W_N is not empty if and only if there exists $\Delta_0 > 0$ such that W_N^Δ is not empty for $\Delta \leq \Delta_0$. If W_N is not empty, then*

$$(42) \quad \lim_{\Delta \rightarrow 0} \rho_H(W_N^\Delta, W_N) = 0$$

and

$$(43) \quad \lim_{\Delta \rightarrow 0} G_N^\Delta = G_N.$$

Also, if γ_N^Δ is a solution of problem (40) and $\lim_{\Delta' \rightarrow 0} \rho(\gamma_N^{\Delta'}, \gamma_N) = 0$ for some sequence of Δ' tending to zero, then γ_N is a solution of (30). If the solution γ_N of problem (30) is unique, then, for any solution γ_N^Δ of (40), $\lim_{\Delta \rightarrow 0} \rho(\gamma_N^\Delta, \gamma_N) = 0$.

Proof. The proof is in section 6. \square

5. Numerical solution of periodic optimization problems. Let us assume that a solution γ^* of problem (21) is unique and that it is generated by a T -periodic admissible pair $(u^{\gamma^*}(\cdot), y^{\gamma^*}(\cdot))$ (see Remark 3 about these assumptions below). Note that, due to Corollary 4(ii), this pair will be a solution of the periodic optimization problem (5). Let

$$(44) \quad \Theta \stackrel{\text{def}}{=} \{(u, y) : (u, y) = (u^{\gamma^*}(\tau), y^{\gamma^*}(\tau)) \text{ for some } \tau \in [0, T]\}.$$

This set can be considered as the graph of the optimal feedback control function, which is defined on the optimal state trajectory $\mathcal{Y} \stackrel{\text{def}}{=} \{y : (u, y) \in \Theta\}$ by the equation $\psi(y) \stackrel{\text{def}}{=} u \quad \forall (u, y) \in \Theta$. For the definition of $\psi(\cdot)$ to make sense, it is assumed that the set Θ is such that from the fact that $(u', y) \in \Theta$ and $(u'', y) \in \Theta$ it follows that $u' = u''$ (this assumption is satisfied if the closed curve defined by $y^{\gamma^*}(\tau)$, $\tau \in [0, T]$, does not intersect itself).

Let $\gamma_N^\Delta \stackrel{\text{def}}{=} \{\gamma_{l,k}^\Delta\}$ be a basic solution of the finite-dimensional LPP (40), that is, a solution of (40) which is an extreme point of W_N^Δ . Let

$$(45) \quad \Theta_N^\Delta \stackrel{\text{def}}{=} \{(u_l, y_k) : \gamma_{l,k}^\Delta > 0\}, \quad \mathcal{Y}_N^\Delta \stackrel{\text{def}}{=} \{y : (u, y) \in \Theta_N^\Delta\}, \quad \psi_N^\Delta(y) \stackrel{\text{def}}{=} u \quad \forall (u, y) \in \Theta_N^\Delta,$$

where again it is assumed that from the fact that $(u', y) \in \Theta_N^\Delta$ and $(u'', y) \in \Theta_N^\Delta$ it follows that $u' = u''$. Note that the set Θ_N^Δ (and the set \mathcal{Y}_N^Δ) can contain no more than $N + 1$ elements since γ_N^Δ , being a basic solution of the LPP (40), has no more than $N + 1$ positive elements (see, e.g., [18, p. 81]).

The two propositions below establish that the set Θ_N^Δ converges (in the specified sense) to the set Θ , thus leading to the corresponding convergences of \mathcal{Y}_N^Δ to \mathcal{Y} and of $\psi_N^\Delta(y)$ to $\psi(y)$.

We will be using the following notation. B will stand for the open unit ball in \mathbb{R}^{n+m} : $B \stackrel{\text{def}}{=} \{(u, y) : \|(u, y)\| < 1\}$ and, for any $Q \subset U \times Y$, $\gamma_N^\Delta(Q)$ will denote the γ_N^Δ measure of Q : $\gamma_N^\Delta(Q) \stackrel{\text{def}}{=} \sum_{(u_l, y_k) \in Q \cap \Theta_N^\Delta} \gamma_{l,k}^\Delta$.

PROPOSITION 10. *Let Assumptions 1 and 2 be satisfied and let γ^* be the unique solution of (21). Then, corresponding to an arbitrary small $r > 0$ and arbitrary small $\delta > 0$, there exists N_0 such that, for $N \geq N_0$ and $\Delta \leq \Delta_N$ (Δ_N is positive and may depend on N),*

$$(46) \quad \gamma_N^\Delta(\Theta_N^\Delta / (\Theta + rB)) < \delta,$$

$$(47) \quad \Theta_N^{\Delta, \delta} \subset \Theta + rB,$$

where $\Theta_N^{\Delta, \delta} \stackrel{\text{def}}{=} \{(u_l, y_k) : \gamma_{l,k}^\Delta \geq \delta\}$.

Proof. The proof is in section 6. \square

Assumption 3. For any $(u, y) \in \text{cl}\Theta$ (the closure of Θ) and any $r > 0$, the set $B_r(u, y) \stackrel{\text{def}}{=} ((u, y) + rB) \cap (U \times Y)$ has a nonzero γ^* -measure: $\gamma^*(B_r(u, y)) > 0$.

Note that this assumption is satisfied if the optimal control function $u^{\gamma^*}(\cdot) : [0, T] \rightarrow U$ is piecewise continuous and at every discontinuity point τ the value of $u^{\gamma^*}(\tau)$ is equal to either the limit from the left ($u^{\gamma^*}(\tau) = \lim_{\tau' \rightarrow \tau-} u^{\gamma^*}(\tau')$) or the limit from the right ($u^{\gamma^*}(\tau) = \lim_{\tau' \rightarrow \tau+} u^{\gamma^*}(\tau')$).

PROPOSITION 11. *Let the conditions of Proposition 10 and Assumption 3 be satisfied. Then, corresponding to an arbitrary small $r > 0$, there exists N_0 such that, for $N \geq N_0$ and $\Delta \leq \Delta_N$ (Δ_N is positive and may depend on N),*

$$(48) \quad \Theta \subset \Theta_N^\Delta + rB.$$

Proof. The proof is in section 6. \square

Based on the consideration above, one can propose the following steps to construct an approximate solution to the periodic optimization problem (5):

(1) Find a basic solution γ_N^Δ and the optimal value G_N^Δ of the LPP (40) for N large and Δ small enough; the values of N and Δ can be identified as being, respectively, *large enough* and *small enough* if a further increase of N and a reduction of Δ lead only to insignificant changes of the optimal value G_N^Δ and, thus, the latter can be considered to be approximately equal to G^* (see Propositions 7 and 9).

(2) Define $\Theta_N^\Delta, \mathcal{Y}_N^\Delta, \psi_N^\Delta(y)$ as in (45). Note that, as follows from Propositions 10 and 11, if γ^* is the unique solution of (21) and it is generated by a periodic admissible pair, then one can expect that the points of \mathcal{Y}_N^Δ will be concentrated around a closed curve being the optimal state trajectory, while $\psi_N^\Delta(y)$ will give a pointwise approximation to the optimal feedback control.

(3) Extrapolate the function $\psi_N^\Delta(y)$ to some neighborhood of \mathcal{Y}_N^Δ and integrate system (1) starting from an initial point $y(0) \in \mathcal{Y}_N^\Delta$ and using the extrapolation of $\psi_N^\Delta(y)$ as the feedback control. One can expect that, thus, the obtained solution of the system will return to a small vicinity of the starting point $y(0)$ and it will be possible to identify the end point of the integration period, T^Δ , as the moment the solution enters this vicinity.

(4) Adjust the initial condition and/or control to obtain a periodic admissible pair $(u^\Delta(\tau), y^\Delta(\tau))$ defined on the interval $[0, T^\Delta]$. Find the integral

$$\frac{1}{T^\Delta} \int_0^{T^\Delta} g(u^\Delta(\tau), y^\Delta(\tau)) d\tau$$

and compare its value with G_N^Δ . If this value proves to be close to G_N^Δ , then, by Corollary 4(ii), the constructed admissible pair is a “good” approximation to the solution of the periodic optimization problem (5).

Remark 3. Under certain conditions (e.g., under the conditions mentioned in Remark 2), the set of occupational measures generated by periodic regimes is dense in W and $G^* = G_\infty = G_{per}$ (compare with Corollary 4). If this is the case, then the assumption that there exists a solution γ^* of problem (21), which is generated by a periodic admissible pair, is equivalent to the assumption that there exists a solution of the periodic optimization problem (5), and the assumption that γ^* is a unique solution of problem (21) implies that all solutions of (5) generate the same occupational measure (namely, γ^*). Note that these assumptions are difficult to verify and one may attempt to use the above steps to find an approximate solution of (5) without such a verification. If, as the result of executing these steps, a periodic admissible pair that gives the value of the objective function close to G_N^Δ is constructed, then one can consider this pair as an approximate solution to problem (5) and use it, if necessary, for further analysis of the existence and structure of the “exact” solution.

Let us illustrate the construction with the following two examples.

Example 1. Let k and ω be positive parameters such that

$$(49) \quad \omega > 1, \quad k\omega < 1.$$

Consider a differential equation

$$(50) \quad \ddot{x}(\tau) + k\dot{x}(\tau) + \omega^2 x(\tau) = u(\tau),$$

where $x(\tau)$ and $u(\tau)$ are scalars and $u(\tau) \in [-1, 1]$. Via a standard replacement of variables (i.e., $x(\tau) = y_1(\tau)$ and $\dot{x}(\tau) = y_2(\tau)$), equation (50) is reduced to the system of the form (1) with

$$y \stackrel{\text{def}}{=} (y_1, y_2), \quad f(u, y) \stackrel{\text{def}}{=} (y_2, -\omega^2 y_1 - ky_2 + u), \quad U = [-1, 1].$$

Since this system is linear and stable, condition (37) is satisfied and, hence, the system has a forward invariant set $Y^* \subset \mathbb{R}^2$ that is a global attractor for its solutions. The Kalman controllability matrix of the system has rank 2 and, consequently, the interior of Y^* is not empty. Thus, as follows from Remark 2, both Assumptions 1 and 2 are satisfied if Y is such that it contains Y^* (in what follows this is achieved by choosing Y large enough). Also, all periodic and steady state solutions of the system are contained in Y^* , which means that all such solutions are admissible, with the set of steady state admissible pairs (see (6)) being, in this case, equal to

$$M = \left\{ (u, y) : u \in [-1, 1]; y = (y_1, y_2), y_1 = \frac{u}{\omega^2}, y_2 = 0 \right\}.$$

Take

$$(51) \quad g(u, y) \stackrel{\text{def}}{=} u^2 - y_1^2$$

and consider the steady state optimization problem (7). By the first inequality in (49), its solution and the optimal value are $u = 0, y_1 = y_2 = 0, G_{ss} = 0$. It is easy to verify that this steady state solution is not optimal in the corresponding periodic optimization problem (5). To see this, it is enough to consider the $\frac{2\pi}{\omega}$ -periodic admissible pair $(u(\tau), y(\tau))$, with $u(\tau) = \cos(\omega\tau)$ and $y(\tau) = (\frac{1}{\omega k} \sin(\omega\tau), \frac{1}{k} \cos(\omega\tau))$. The value of the objective function obtained on this pair is

$$(52) \quad \tilde{G} \stackrel{\text{def}}{=} \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \left(\cos^2(\omega\tau) - \frac{1}{\omega^2 k^2} \sin^2(\omega\tau) \right) d\tau = \frac{1}{2} \left(1 - \frac{1}{\omega^2 k^2} \right) < 0.$$

The last inequality follows from the second inequality in (49), which postulates smallness of the “friction coefficient” k compared to the “proper frequency” ω and, thus, makes it possible to diminish the value of the objective function via the resonance oscillations of the state variables (such an interpretation of the example was given by Pervozvanskii; see Examples 3.2 and 4.2 in [24]).

Let us demonstrate numerical results obtained with the use of the proposed linear programming approach for the case when $k = 0.3$ and $\omega = 2$. Note that, for these values of the parameters, $\tilde{G} = \frac{1}{2} \left(1 - \frac{1}{0.36} \right) \approx -0.889$.

Let us take $Y \stackrel{\text{def}}{=} \{(y_1, y_2) \mid y_i \in [-5, 5], i = 1, 2\}$ (it is straightforward to verify that $Y^* \subset Y$ in this case) and define

$$(53) \quad u_i \stackrel{\text{def}}{=} -1 + i\Delta, \quad y_{1,j} \stackrel{\text{def}}{=} -5 + j\Delta, \quad y_{2,k} \stackrel{\text{def}}{=} -5 + k\Delta,$$

where $i = 0, 1, \dots, \frac{2}{\Delta}$ and $j, k = 0, 1, \dots, \frac{10}{\Delta}$ (Δ being chosen in such a way that $\frac{2}{\Delta}$ is integer). Using a slightly different system of notation (adjusted to the case under

consideration and to the grid defined by (53)) and using monomials as the functions defining the constraints in (39), one can rewrite the LPP (40) in the form

$$(54) \quad \min_{\gamma \in W_N^\Delta} \sum_{i,j,k} ((u_i)^2 - (y_{1,j})^2) \gamma_{i,j,k} \stackrel{\text{def}}{=} G_N^\Delta,$$

with

$$(55) \quad W_N^\Delta \stackrel{\text{def}}{=} \left\{ \gamma = \{\gamma_{i,j,k}\} \geq 0 : \sum_{i,j,k} \gamma_{i,j,k} = 1, \sum_{i,j,k} (\phi'_{l_1,l_2}(y_{1,j}, y_{2,k}))^T f(u_i, y_{1,j}, y_{2,k}) \gamma_{i,j,k} = 0, \right. \\ \left. l_1, l_2 = 0, 1, \dots, N \right\},$$

where $\phi_{l_1,l_2}(y_1, y_2) \stackrel{\text{def}}{=} y_1^{l_1} y_2^{l_2}$. Problem (54) was solved for $N = 7$ and $N = 10$ and for $\Delta = 0.2, 0.1, 0.05, 0.025, 0.0125$ (note that in both Examples 1 and 2 that follow we used ILOG CPLEX 8.0. (<http://www.ilog.com>) as a linear programming solver). The optimal values of the LPPs obtained with these values of the parameters are (respectively) $G_7^{0.2} \approx -1.312$, $G_7^{0.1} \approx -1.329$, $G_7^{0.05} \approx -1.331$, $G_7^{0.025} \approx -1.331$, $G_7^{0.0125} \approx -1.331$, and $G_{10}^{0.2} \approx -1.280$; $G_{10}^{0.1} \approx -1.325$, $G_{10}^{0.05} \approx -1.326$, $G_{10}^{0.025} \approx -1.327$, $G_{10}^{0.0125} \approx -1.327$. On the basis of these results and Proposition 9 (see (42)) one may conclude that $G_{10} = \lim_{\Delta \rightarrow 0} G_{10}^\Delta \approx -1.327$. Hence, by (31), $G^* \geq -1.327$ (within the given proximity). From Corollary 4(ii) (see also (8) and Corollary 3) it now follows that, if for some admissible T -periodical pair $(u(\tau), y(\tau))$,

$$(56) \quad \frac{1}{T} \int_0^T (u^2(\tau) - y_1^2(\tau)) d\tau \approx -1.327,$$

then this pair is an approximate solution of problems (3) and (5).

Let $\{\gamma_{i,j,k}^{N,\Delta}\}$ stand for the solution of problem (54). The sets Θ_N^Δ and \mathcal{Y}_N^Δ can then be represented in the form

$$\Theta_N^\Delta = \{(u_i, y_{1,j}, y_{2,k}) : \gamma_{i,j,k}^{N,\Delta} \neq 0\}, \quad \mathcal{Y}_N^\Delta = \left\{ (y_{1,j}, y_{2,k}) : \sum_i \gamma_{i,j,k}^{N,\Delta} \neq 0 \right\}.$$

Let us mark with dots the points on the plane (y_1, y_2) which belong to \mathcal{Y}_N^Δ for $N = 10$ and $\Delta = 0.0125$. The result of such marking is depicted in Figure 1.

Figure 1 clearly identifies a closed curve, which one can expect to be an approximation to the optimal state trajectory. As can be seen from this figure, the value of y_1 is uniquely determined by the value of y_2 for $(y_1, y_2) \in \mathcal{Y}_N^\Delta$ with $y_1 \geq 0$ and for $(y_1, y_2) \in \mathcal{Y}_N^\Delta$ with $y_1 < 0$. Having this in mind, let us mark with dots the points $(u_i, y_{2,k})$ on the plane (u, y_2) for which $\gamma_{i,j,k}^{N,\Delta} \neq 0$ and $y_{1,j} \geq 0$ (Figure 2) and, also, the points for which $\gamma_{i,j,k}^{N,\Delta} \neq 0$ and $y_{1,j} < 0$ (Figure 3).

The points marked with the dots in Figure 2 define $\psi_N^\Delta(y) \stackrel{\text{def}}{=} \psi_N^\Delta(y_1, y_2)$ as a function of y_2 (denoted as $\bar{\psi}_N^\Delta(y_2)$) for $y_1 \geq 0$, and the points marked with the dots in Figure 3 define $\psi_N^\Delta(y) \stackrel{\text{def}}{=} \psi_N^\Delta(y_1, y_2)$ as another function of y_2 (denoted as $\bar{\bar{\psi}}_N^\Delta(y_2)$) for $y_1 < 0$. Note that in both cases $(y_1, y_2) \in \mathcal{Y}_N^\Delta$. Let us extend the definition of

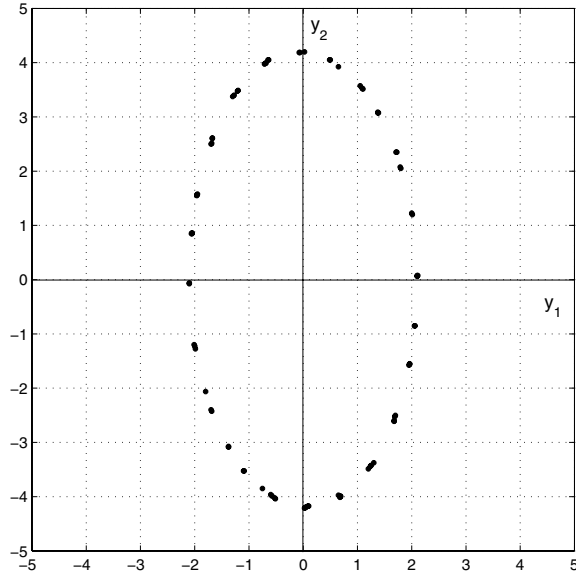


FIG. 1. \mathcal{Y}_N^Δ .

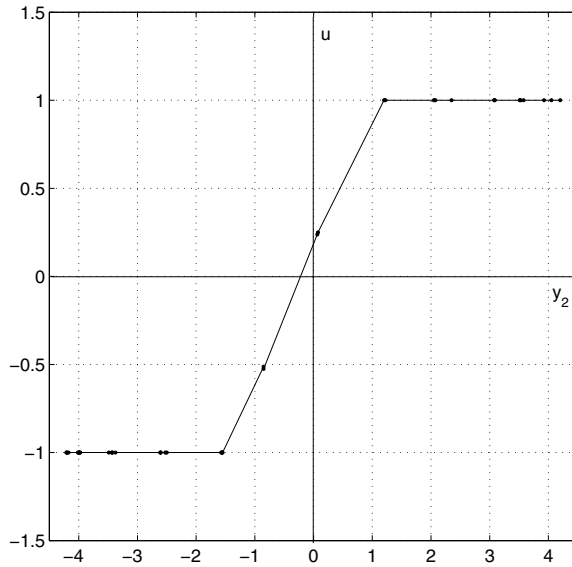


FIG. 2. $\psi_N^\Delta(y_1, y_2) = \bar{\psi}_N^\Delta(y_2)$ for $y_1 \geq 0$ and $(y_1, y_2) \in \mathcal{Y}_N^\Delta$.

$\psi_N^\Delta(y_1, y_2)$ by connecting the points in Figures 2 and 3 with piecewise linear functions (we will denote this extension also as $\psi_N^\Delta(y_1, y_2)$) and integrate the system with the feedback control thus obtained and with the initial condition being at one of the points marked in Figure 1. Denote by $\tilde{y}^\Delta(\tau) = (\tilde{y}_1^\Delta(\tau), \tilde{y}_2^\Delta(\tau))$ the resulting solution of the system and by $u^\Delta(\tau) = \psi_N^\Delta(\tilde{y}_1^\Delta(\tau), \tilde{y}_2^\Delta(\tau))$ the corresponding open loop control. The function $\tilde{y}^\Delta(\tau)$ proves to be nonperiodic but it returns to a small vicinity of $\tilde{y}^\Delta(0)$

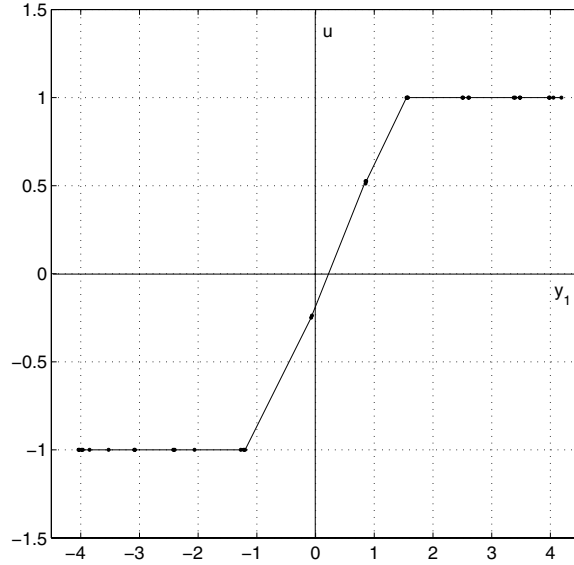


FIG. 3. $\psi_N^\Delta(y_1, y_2) = \bar{\psi}_N^\Delta(y_2)$ for $y_1 < 0$ and $(y_1, y_2) \in \mathcal{Y}_N^\Delta$.

with $\tau \approx 3.16$. Take $T^\Delta \stackrel{\text{def}}{=} 3.16$ and denote by $y^\Delta(\tau)$ the solution of the system which is obtained when applying the control $u^\Delta(\tau)$ on the interval $[0, T^\Delta]$ and which satisfies the periodicity condition $y^\Delta(0) = y^\Delta(T^\Delta)$. Note that such a solution exists, it is unique and, for the system under consideration, it can be easily found numerically (see, e.g., [23, p. 39]). The periodic admissible pair $(u^\Delta(\tau), y^\Delta(\tau))$ that has been constructed by following the indicated steps is shown in Figures 4 and 5.

The value of the objective function calculated on this pair is approximately equal to -1.324 . Comparing it with (56), one can see that it is close to the optimal one and, hence, the pair $(u^\Delta(\tau), y^\Delta(\tau))$ can be considered to be an approximate solution to problem (5). Let us emphasize that we have not verified the assumptions that the solution of the periodic optimization problem (5) exists and that it is unique. Based on the form of the obtained approximate solution, one may conjecture that these assumptions are satisfied in the given example.

Example 2. Consider system (1) with

$$y \stackrel{\text{def}}{=} (y_1, y_2), \quad u \stackrel{\text{def}}{=} (u_1, u_2), \quad f(u, y) \stackrel{\text{def}}{=} (-y_1 + u_1, -y_2 + u_2)$$

(that is, $n = 2$ and $m = 2$) and with

$$U \stackrel{\text{def}}{=} [-1, 1] \times [-1, 1], \quad Y \stackrel{\text{def}}{=} [-1, 1] \times [-1, 1].$$

As in Example 1, the system under consideration is linear and stable, and, hence, it has a forward invariant set Y^* which is also a global attractor of its solutions. Moreover, it can be easily verified that Y^* coincides with Y introduced above. This implies that both Assumptions 1 and 2 are satisfied (see Remarks 1 and 2).

Let the function $g(u, y)$ be defined by the equation

$$(57) \quad g(u, y) \stackrel{\text{def}}{=} -y_1 u_2 + y_2 u_1$$

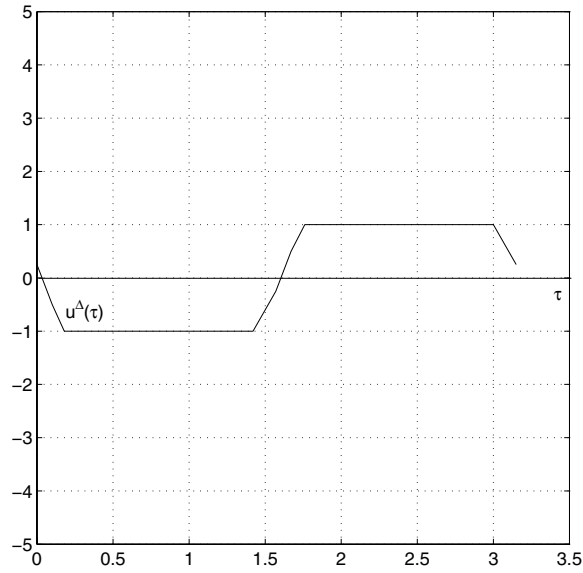


FIG. 4. $u^\Delta(\tau)$.

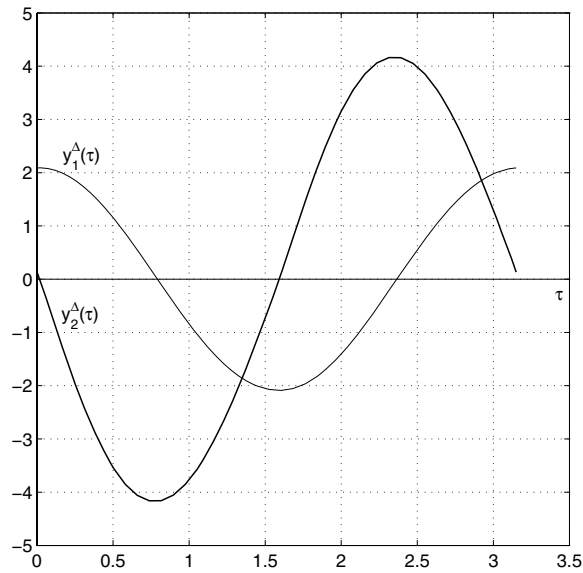


FIG. 5. $y_1^\Delta(\tau)$ and $y_2^\Delta(\tau)$.

and let

$$(58) \quad u_{1,i}^\Delta \stackrel{\text{def}}{=} -1 + i\Delta, \quad u_{2,l}^\Delta \stackrel{\text{def}}{=} -1 + l\Delta, \quad y_{1,j}^\Delta \stackrel{\text{def}}{=} -1 + j\Delta, \quad y_{2,k}^\Delta \stackrel{\text{def}}{=} -1 + k\Delta,$$

where $i, l, j, k = 0, 1, \dots, \frac{2}{\Delta}$ (Δ is such that $\frac{2}{\Delta}$ is integer). The LPP (40) takes the

form

$$(59) \quad \min_{\gamma \in W_N^\Delta} \sum_{i,l,j,k} (-y_{1,j}^\Delta u_{2,l}^\Delta + y_{2,k}^\Delta u_{1,i}^\Delta) \gamma_{i,l,j,k} \stackrel{\text{def}}{=} G_N^\Delta,$$

where

$$(60) \quad W_N^\Delta \stackrel{\text{def}}{=} \left\{ \gamma = \{\gamma_{i,l,j,k}\} \geq 0 : \sum_{i,l,j,k} \gamma_{i,l,j,k} = 1, \right. \\ \left. \sum_{i,l,j,k} (\phi'_{l_1,l_2}(y_{1,j}, y_{2,k}))^T f(u_{1,i}, u_{2,l}, y_{1,j}, y_{2,k}) \gamma_{i,l,j,k} = 0, l_1, l_2 = 0, 1, \dots, N \right\},$$

with $\phi_{l_1,l_2}(y_1, y_2) \stackrel{\text{def}}{=} y_1^{l_1} y_2^{l_2}$.

Problem (59) was solved for $N = 10$ and for $\Delta = 0.2, 0.1, 0.05, 0.025, 0.0125, 0.00625$, and 0.003125 . The optimal values of the LPPs obtained with these parameters are $G_{10}^{0.2} \approx -0.7035$, $G_{10}^{0.1} \approx -0.7579$, $G_{10}^{0.05} \approx -0.7671$, $G_{10}^{0.025} \approx -0.7678$, $G_{10}^{0.0125} \approx -0.7679$, $G_{10}^{0.00625} \approx -0.7679$, $G_{10}^{0.003125} \approx -0.7679$.

Similarly to Example 1, one may conclude that $G^* \geq G_{10} = \lim_{\Delta \rightarrow 0} G_{10}^\Delta \approx -0.7679$. Consequently, by Corollary 4(ii), if for some admissible T -periodic pair $(u(\tau), y(\tau))$,

$$(61) \quad \frac{1}{T} \int_0^T (-y_1(\tau)u_2(\tau) + y_2(\tau)u_1(\tau))d\tau \approx -0.7679,$$

then this pair is an approximate solution of the periodic optimization problem under consideration.

Let $\{\gamma_{i,l,j,k}^{N,\Delta}\}$ be the solution of problem (59). Then

$$\Theta_N^\Delta = \{(u_{1,i}^\Delta, u_{2,l}^\Delta, y_{1,j}^\Delta, y_{2,k}^\Delta) : \gamma_{i,l,j,k}^{N,\Delta} \neq 0\}, \quad \mathcal{Y}_N^\Delta = \left\{ (y_{1,j}^\Delta, y_{2,k}^\Delta) : \sum_{i,l} \gamma_{i,l,j,k}^{N,\Delta} \neq 0 \right\}.$$

Figure 6 represents the result of marking with dots the points on the plane (y_1, y_2) which belong to \mathcal{Y}_N^Δ for $N = 10$ and $\Delta = 0.003125$.

The image created by the points marked in Figure 6 reminds a square. The analysis of the results of the linear programming solution showed that the function $\psi_N^\Delta(y) \stackrel{\text{def}}{=} \psi_N^\Delta(y_1, y_2)$ is equal to $(-1, 1)$ at every point belonging to the upper side of the “square,” and it is equal to $(-1, -1)$, $(1, -1)$, and $(1, 1)$ at the points belonging to, respectively, left, bottom, and right sides of the square. Let us extend the definition of $\psi_N^\Delta(y)$ as follows:

- $u_1 = -1, u_2 = 1$ for $-0.5 < y_1 \leq 0.9, 0.5 \leq y_2 \leq 0.9$;
- $u_1 = -1, u_2 = -1$ for $-0.9 \leq y_1 \leq -0.5, -0.5 < y_2 \leq 0.9$;
- $u_1 = 1, u_2 = -1$ for $-0.9 \leq y_1 < 0.5, -0.9 \leq y_2 \leq 0.5$;
- $u_1 = 1, u_2 = 1$ for $0.5 \leq y_1 \leq 0.9, -0.9 \leq y_2 < 0.5$.

Proceeding as in Example 1, we integrate the system with thus defined feedback control and with the initial condition being at one of the points marked in Figure 6. The resulting solution of the system $\tilde{y}^\Delta(\tau) = (\tilde{y}_1^\Delta(\tau), \tilde{y}_2^\Delta(\tau))$ remains in the area of definition of the feedback control and it returns to a small vicinity of $\tilde{y}^\Delta(0)$ with $\tau \approx 6.1$.

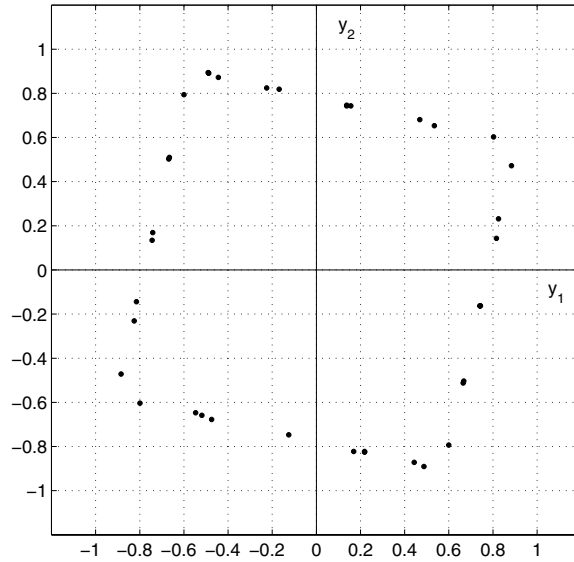


FIG. 6. \mathcal{Y}_N^Δ .

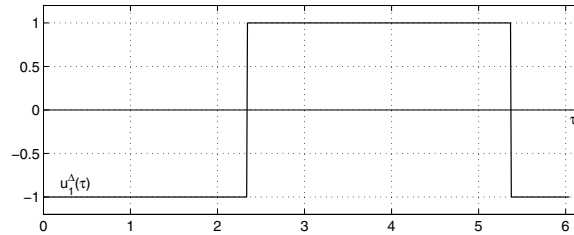


FIG. 7. $u_1^\Delta(\tau)$.

Take $T^\Delta \stackrel{\text{def}}{=} 6.1$. Let $u^\Delta(\tau) = (u_1^\Delta(\tau), u_2^\Delta(\tau))$ be the open loop control defined by the feedback control on the trajectory specified above and let $y^\Delta(\tau) = (y_1^\Delta(\tau), y_2^\Delta(\tau))$ be the solution of the system obtained with this open loop control which satisfies the periodicity condition: $y^\Delta(0) = y^\Delta(T^\Delta)$. The components of the constructed periodic admissible pair $(u^\Delta(\tau), y^\Delta(\tau))$ are shown in Figures 7, 8, and 9.

The value of the objective function calculated on the pair $(u^\Delta(\tau), y^\Delta(\tau))$ is approximately equal to -0.7679 and, hence, this pair is an approximate solution to problem (5). Note that in this example too the assumptions that the solution of the periodic optimization problem (5) exists and that it is unique have not been verified. However, again, based on the form of the obtained approximate solution, one may conjecture that the solution exists and that it has a form similar to that of the obtained approximate solution.

Example 2 (continued). The set of steady state admissible pairs in Example 2 is equal to

$$M = \{(u, y) : u = (u_1, u_2), \quad y = (y_1, y_2), \quad y_i = u_i \in [0, 1], \quad i = 1, 2\}.$$

One can see that, for every $(u, y) \in M$, $g(u, y) = 0$. Hence, $G_{ss} = 0 < G_{per} = G^* \approx -0.7679$. Consider the periodic optimization problem (5) in which $g(u, y)$ is replaced

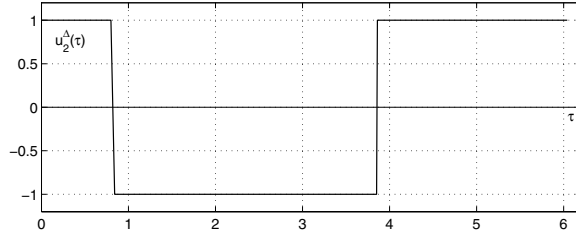


FIG. 8. $u_2^\Delta(\tau)$.

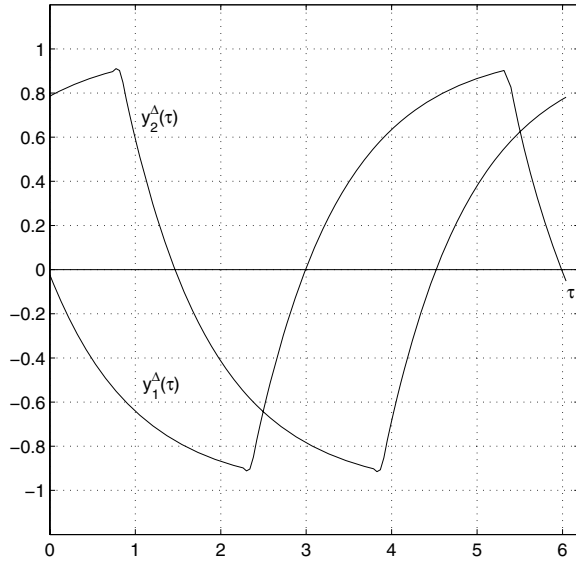


FIG. 9. $y_1^\Delta(\tau)$ and $y_2^\Delta(\tau)$.

by $g^\lambda(u, y)$,

$$(62) \quad g^\lambda(u, y) \stackrel{\text{def}}{=} -y_1 u_2 + y_2 u_1 + \lambda(u_1^2 + u_2^2 + y_1^2 + y_2^2),$$

where $\lambda \geq 0$. For every $(u, y) \in M$, $g^\lambda(u, y) = \lambda(u_1^2 + u_2^2 + y_1^2 + y_2^2)$. Consequently, the optimal value of the corresponding steady state optimization problem (denoted G_{ss}^λ) is equal to zero too: $G_{ss}^\lambda = 0 \ \forall \lambda \geq 0$. It is well known (see, e.g., [23]) that, if U and Y are convex, the system is linear, and the integrand in the objective function is convex, then the optimal values of the periodic and steady state optimization problems coincide. One can verify (by direct calculation) that the Hessian of the function $g^\lambda(u, y)$ has nonnegative eigenvalues for $\lambda \geq 0.5$. That is, this function is convex and, hence, $G_p^\lambda = G_{ss}^\lambda = 0 \ \forall \lambda \geq 0.5$, where G_p^λ stands for the optimal value of the periodic optimization problem (5) considered with $g^\lambda(u, y)$ instead of $g(u, y)$. Consider the LPP

$$(63) \quad \min_{\gamma \in W_N^\Delta} \sum_{i,l,j,k} (-y_{1,j}^\Delta u_{2,l}^\Delta + y_{2,k}^\Delta u_{1,i}^\Delta + \lambda((u_{1,i}^\Delta)^2 + (u_{2,l}^\Delta)^2 + (y_{1,j}^\Delta)^2 + (y_{2,k}^\Delta)^2)) \gamma_{i,l,j,k} \stackrel{\text{def}}{=} G_N^{\Delta,\lambda},$$

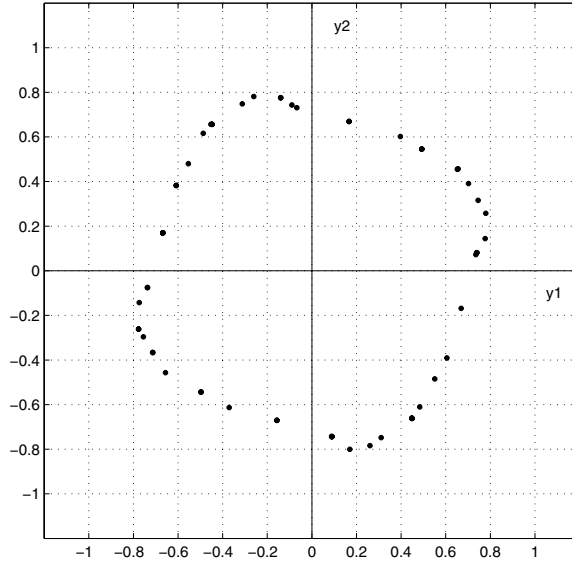


FIG. 10. \mathcal{Y}_N^Δ for $\lambda = 0.2$.

where W_N^Δ is as in (60). As above, the solution of this problem with $N = 10$ and $\Delta = 0.003125$ allows one to find an approximation to the solution of the corresponding periodic optimization problem for different λ . In particular, for $\lambda = 0.2, 0.33, 0.35$, the approximations to the optimal values of the latter are, respectively, $G_{per}^{0.2} \approx -0.2767$, $G_{per}^{0.33} \approx -0.0358$, $G_{per}^{0.35} \approx -0.0050$. For $\lambda \geq 0.36$, $G_{per}^\lambda \approx G_{ss}^\lambda = 0$ (for $\lambda \geq 0.5$, this being due to the convexity of the function $g^\lambda(u, y)$). Figures 10–13, represent the results of marking with dots the points on the plane (y_1, y_2) which belong to \mathcal{Y}_N^Δ for $\lambda = 0.2, 0.33, 0.35, 0.36$ ($N = 10$, $\Delta = 0.003125$).

6. Proofs for sections 3, 4, and 5.

Proof of Proposition 2. To prove the validity of (20), let us define $\kappa(S)$ by the equation

$$(64) \quad \kappa(S) \stackrel{\text{def}}{=} \sup_{\gamma \in \Gamma(S)} \rho(\gamma, W)$$

and show that $\kappa(S)$ tends to zero as S tends to infinity. Assume it is not the case. Then there exist a positive number δ and sequences $S^k \rightarrow \infty$, $\gamma^k \in \Gamma(S^k)$, such that $\rho(\gamma^k, W) \geq \delta$ for $k = 1, 2, \dots$. Without loss of generality one may assume that there exists $\lim_{k \rightarrow \infty} \gamma^k \stackrel{\text{def}}{=} \gamma \in \mathcal{P}(U \times Y)$ (since $\mathcal{P}(U \times Y)$ is compact). From the continuity of the metric it follows that

$$(65) \quad \rho(\gamma, W) \geq \delta.$$

By the definition of the convergence in $\mathcal{P}(U \times Y)$ (see (16)),

$$(66) \quad \lim_{k \rightarrow \infty} \int_{U \times Y} (\phi'(y))^T f(u, y) \gamma^k(du, dy) = \int_{U \times Y} (\phi'(y))^T f(u, y) \gamma(du, dy)$$

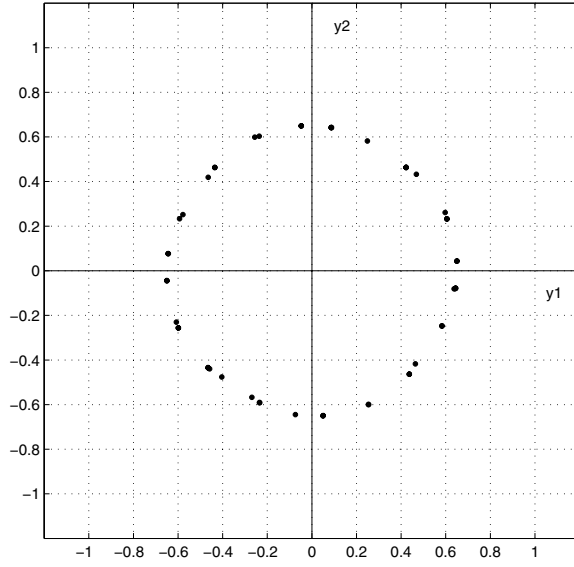


FIG. 11. \mathcal{Y}_N^Δ for $\lambda = 0.33$.

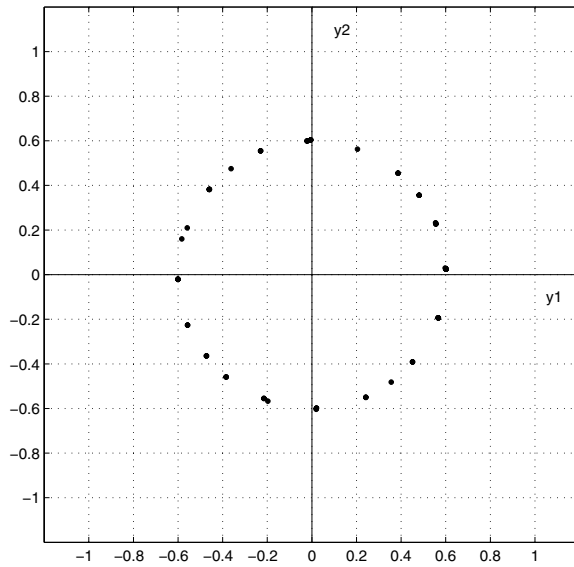


FIG. 12. \mathcal{Y}_N^Δ for $\lambda = 0.35$.

for any $\phi \in C^1$. Also, from the fact that $\gamma^k \in \Gamma(S^k)$ it follows that there exists an admissible pair $(u^k(\tau), y^k(\tau))$ defined on the interval $[0, S^k]$ such that

$$\int_{U \times Y} (\phi'(y))^T f(u, y) \gamma^k(du, dy) = \frac{1}{S^k} \int_0^{S^k} (\phi'(y^k(\tau)))^T f(u^k(\tau), y^k(\tau)) d\tau.$$

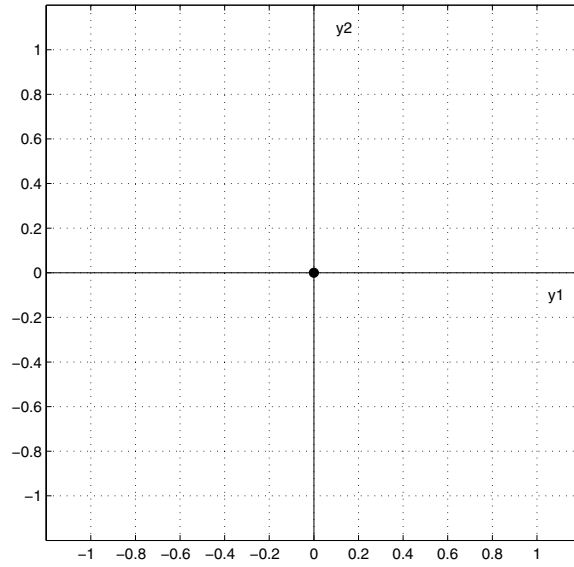


FIG. 13. \mathcal{Y}_N^Δ for $\lambda = 0.36$.

The second integral is apparently equal to

$$\frac{\phi(y^k(S^k)) - \phi(y^k(0))}{S^k}$$

and tends to zero as S^k tends to infinity (since $y^k(\tau) \in Y \forall \tau \in [0, S^k]$ and Y is a compact set). This and (66) imply that

$$\int_{U \times Y} (\phi'(y))^T f(u, y) \gamma(du, dy) = 0 \forall \phi \in C^1 \Rightarrow \gamma \in W.$$

The latter contradicts (65) and, hence, $\kappa(S)$ defined in (64) tends to zero as S tends to infinity. This proves (20).

From the consideration above it follows that, if there exists a sequence of S^k tending to infinity such that $\Gamma(S^k) \neq \emptyset$, then the set W is not empty. Hence, if the latter is empty, then $\Gamma(S) = \emptyset$ for all S large enough. \square

Proof of Proposition 5. Let \hat{Y} be a compact set which contains Y in its interior and let $q_l(u, y) : U \times \hat{Y} \rightarrow \mathbb{R}^1, l = 1, 2, \dots$, be a sequence of Lipschitz continuous functions which is dense in $C(U \times \hat{Y})$ (the space of continuous functions on $U \times \hat{Y}$). Let

$$(67) \quad h(u, y) = (q_1(u, y), \dots, q_j(u, y)), \quad \bar{h}(\nu, y) = (\bar{q}_1(\nu, y), \dots, \bar{q}_j(\nu, y)), \quad j = 1, 2, \dots,$$

where $\bar{q}_j(\nu, u) \stackrel{\text{def}}{=} \int_U q_j(u, y) \nu(du)$. Define the sets $V_h(S, y)$ and $\bar{V}_h(S, y)$ by the equations

$$V_h(S) \stackrel{\text{def}}{=} \bigcup_{(u(\cdot), y(\cdot))} \left\{ \frac{1}{S} \int_0^S h(u(\tau), y(\tau)) d\tau \right\},$$

$$\bar{V}_h(S) \stackrel{\text{def}}{=} \bigcup_{(\nu(\cdot), y(\cdot))} \left\{ \frac{1}{S} \int_0^S \bar{h}(\nu(\tau), y(\tau)) d\tau \right\},$$

where the unions are over all admissible and relaxed admissible pairs, respectively.

Let d_H stand for the Hausdorff metric defined on compact subsets of \mathbb{R}^j by the Euclidean norm. Using a standard argument based on the separability of convex sets (see, e.g., Lemma 4.2 in [26]), one can verify that Assumption 1 is equivalent to the statement that, for any $h(\cdot)$ and $\bar{h}(\cdot)$ as in (67),

$$(68) \quad d_H(\bar{c}oV_h(S), \bar{c}o\bar{V}_h(S)) \stackrel{\text{def}}{=} \kappa_h(S) \rightarrow 0$$

as $S \rightarrow \infty$, where $\bar{c}o$ stands for the closed convex hull of the corresponding set.

From the definition of the metric ρ in the form (15) and the convexity of W it follows that $\sup_{\gamma \in \Gamma(S)} \rho(\gamma, W) = \sup_{\gamma \in co\Gamma(S)} \rho(\gamma, W)$. Hence, by (20),

$$\lim_{S \rightarrow \infty} \sup_{\gamma \in co\Gamma(S)} \rho(\gamma, W) = 0$$

and, to prove (24), it is enough to show that

$$(69) \quad \sup_{\gamma \in W} \rho(\gamma, co\Gamma(S)) \leq \kappa(S), \quad \lim_{S \rightarrow \infty} \kappa(S) = 0.$$

Let us take an arbitrary $\gamma \in W$. From Lemma 5.1 in [26] and Theorem 4.1 in [43] it follows (see [26]) that there exist a probability space (Ω, \mathcal{F}, P) and a $(\mathcal{P}(U) \times \mathbb{R}^m)$ -valued random process $(\nu(\tau), y(\tau)) = (\nu(\tau, \omega), y(\tau, \omega))$ such that (i) for any $h(\cdot)$ and $\bar{h}(\cdot)$ defined in (67),

$$(70) \quad E[\bar{h}(\nu(\tau, \omega), y(\tau, \omega))] = \int_{U \times Y} h(u, y) \gamma(du, dy) \quad \forall \tau \geq 0;$$

and (ii) for some $\Omega' \subset \Omega$ with $P(\Omega') = 1$ and for any $\omega \in \Omega'$, the pair $(\nu(\cdot, \omega), y(\cdot, \omega))$ is relaxed admissible on any interval $[0, S]$.

From (ii) it follows that

$$\frac{1}{S} \int_0^S \bar{h}(\nu(\tau, \omega), y(\tau, \omega)) d\tau \in \bar{V}_h(S) \quad \forall \omega \in \Omega',$$

while (i) implies that

$$\int_{U \times Y} h(u, y) \gamma(du, dy) = E \frac{1}{S} \int_0^S [\bar{h}(\nu(\tau, \omega), y(\tau, \omega))] d\tau \in \bar{c}o\bar{V}_h(S).$$

Using the above inclusion and taking into account (68) (as well as the fact that γ is an arbitrary element of W), one can conclude that

$$(71) \quad \bigcup_{\gamma \in W} \left\{ \int_{U \times Y} h(u, y) \gamma(du, dy) \right\} \subset \bar{c}oV_h(S) + \kappa_h(S) B_j,$$

where B_j is the closed unit ball in \mathbb{R}^j (j is the number of components of $h(\cdot)$; see (67)). Applying now Lemma 3.5 from [27] in exactly the same way as it is done

on page 335 in [26], one can prove the validity of (69). This completes the proof of (24). \square

Proof of Proposition 7 (continued). Since $W \subset W_N$, to prove that (32) is valid, it is enough to show that

$$(72) \quad \lim_{N \rightarrow \infty} \sup_{\gamma \in W_N} \rho(\gamma, W) = 0.$$

Assume it is not true. Then there exist a positive number δ , a subsequence of positive integers $N' \rightarrow \infty$, and a sequence of probability measures $\gamma_{N'} \in W_{N'}$ such that $\rho(\gamma_{N'}, W) \geq \delta$. Due to the compactness of $\mathcal{P}(U \times Y)$, one may assume (without loss of generality) that there exists $\bar{\gamma} \in \mathcal{P}(U \times Y)$ such that

$$(73) \quad \lim_{N' \rightarrow \infty} \rho(\gamma_{N'}, \bar{\gamma}) = 0 \quad \Rightarrow \quad \rho(\bar{\gamma}, W) \geq \delta.$$

From the fact that $\gamma_{N'} \in W_{N'}$ it follows that, for any integer i and $N' \geq i$,

$$\int_{U \times Y} (\phi'_i(y))^T f(u, y) \gamma_{N'}(du, dy) = 0 \quad \Rightarrow \quad \int_{U \times Y} (\phi'_i(y))^T f(u, y) \bar{\gamma}(du, dy) = 0.$$

Since the latter is valid for any $i = 1, 2, \dots$, one can conclude that $\bar{\gamma} \in W$, which contradicts (73). This proves (32). \square

Proof of Proposition 8. In case (i), from (36) it follows that there exist an open ball $B \subset Y$ centered at \bar{y} and a number $r > 0$ such that the closed ball \bar{B}_r centered at 0 and having the radius $r > 0$ will satisfy the inclusion

$$(74) \quad \bar{B}_r \subset f(y, U) \quad \forall y \in B.$$

Let us show that, if $\phi(\cdot)$ satisfies (35), then $\phi'(y) = 0 \ \forall y \in B$ and, thus, Y^* can be taken to be equal to the closure of B . Assume that $\phi'(y) \neq 0$ for some $y \in B$. By (74), there exist $u \in U$ such that $\frac{r}{\|\phi'(y)\|} \phi'(y) = f(y, u)$. Hence, by (35),

$$(\phi'(y))^T \phi'(y) = \frac{\|\phi'(y)\|}{r} (\phi'(y))^T f(y, u) \leq 0 \quad \Rightarrow \quad \phi'(y) = 0.$$

The obtained contradiction proves the statement.

In case (ii), let us show that Y^* can be taken to be equal to the closure of Y^0 . To show this, it is enough to establish that from (35) it follows that $\phi(y) = \text{const} \ \forall y \in Y^0$ (which leads to that $\phi(y) = \text{const} \ \forall y \in Y^*$ and, hence, to that $\phi'(y) = 0 \ \forall y \in \text{int} \ Y^*$).

Let $y', y'' \in Y^0$ and $(u(\tau), y(\tau))$ be an admissible pair such that $y(0) = y'$ and $y(S) = y''$. Then, by (35),

$$\phi(y'') - \phi(y') = \int_0^S (\phi'(y(\tau)))^T f(u(\tau), y(\tau)) d\tau \leq 0 \quad \Rightarrow \quad \phi(y'') \leq \phi(y').$$

Since y', y'' are arbitrary points in Y^0 , the latter implies that

$$\phi(y) = \text{const} \ \forall y \in Y^0. \quad \square$$

Proof of Proposition 9. First, note that, by (41), the set W_N is not empty if W_N^Δ is not empty.

Let us assume that the set W_N is not empty and show that W_N^Δ is not empty and that (42) is valid (the validity of (43) follows from (42) on the basis of Lemma 1(ii);

the other statements included in the proposition are immediate consequences of (42) and (43)).

From (38) and the fact that the functions $(\phi'_i(y))^T f(u, y)$ are continuous it follows that

$$(75) \quad \sup_{(u,y) \in Q_{l,k}^\Delta} |(\phi'_i(y))^T f(u, y) - (\phi'_i(y_k))^T f(u_l, y_k)| \leq \kappa(\Delta), \quad i = 1, \dots, N,$$

for some $\kappa(\Delta)$ such that $\lim_{\Delta \rightarrow 0} \kappa(\Delta) = 0$. Define the set $Z_N^\Delta \subset \mathbb{R}^{L^\Delta + K^\Delta}$ by the equation

$$(76) \quad Z_N^\Delta \stackrel{\text{def}}{=} \left\{ \gamma = \{\gamma_{l,k}\} \geq 0 : \sum_{l,k} \gamma_{l,k} = 1, \left| \sum_{l,k} (\phi'_i(y_k))^T f(u_l, y_k) \gamma_{l,k} \right| \leq \kappa(\Delta), \right. \\ \left. i = 1, 2, \dots, N \right\}.$$

For any Δ , let $\gamma^\Delta \in W_N$ be such that $\rho(\gamma^\Delta, Z_N^\Delta) = \max_{\gamma \in W_N} \rho(\gamma, Z_N^\Delta)$ (γ^Δ exists since W_N is compact) and show that

$$(77) \quad \lim_{\Delta \rightarrow 0} \max_{\gamma \in W_N} \rho(\gamma, Z_N^\Delta) = \lim_{\Delta \rightarrow 0} \rho(\gamma^\Delta, Z_N^\Delta) = 0.$$

Let $\gamma_{l,k}^\Delta \stackrel{\text{def}}{=} \int_{Q_{l,k}^\Delta} \gamma^\Delta(du, dy)$. By (75),

$$\left| \sum_{l,k} (\phi'_i(y_k))^T f(u_l, y_k) \gamma_{l,k}^\Delta \right| \\ = \left| \sum_{l,k} (\phi'_i(y_k))^T f(u_l, y_k) \gamma_{l,k}^\Delta - \int_{U \times Y} (\phi'_i(y))^T f(u, y) \gamma^\Delta(du, dy) \right| \\ \leq \sum_{l,k} \int_{Q_{l,k}^\Delta} |(\phi'_i(y_k))^T f(u_l, y_k) - (\phi'_i(y))^T f(u, y)| \gamma^\Delta(du, dy) \leq \kappa(\Delta), \quad i = 1, 2, \dots, N.$$

Hence, denoting $\tilde{\gamma}^\Delta \stackrel{\text{def}}{=} (\gamma_{l,k}^\Delta)$, one obtains that $\tilde{\gamma}^\Delta \in Z_N^\Delta$ and, consequently,

$$(78) \quad \rho(\tilde{\gamma}^\Delta, Z_N^\Delta) = 0.$$

Let $q(u, y) : U \times Y \rightarrow \mathbb{R}^1$ be an arbitrary continuous function and let $\kappa_q(\Delta)$ be such that

$$\sup_{(u,y) \in Q_{l,k}^\Delta} |q(u, y) - q(u_l, y_k)| \leq \kappa_q(\Delta), \quad \lim_{\Delta \rightarrow 0} \kappa_q(\Delta) = 0.$$

Then

$$\left| \int_{U \times Y} q(u, y) \gamma^\Delta(du, dy) - \sum_{l,k} q(u_l, y_k) \gamma_{l,k}^\Delta \right| \\ = \left| \sum_{l,k} \int_{Q_{l,k}^\Delta} q(u, y) \gamma^\Delta(du, dy) - \sum_{l,k} \int_{Q_{l,k}^\Delta} q(u_l, y_k) \gamma^\Delta(du, dy) \right| \leq \kappa_q(\Delta).$$

The fact that the latter inequality is valid for an arbitrary continuous $q(u, y)$ implies that $\lim_{\Delta \rightarrow 0} \rho(\gamma^\Delta, \tilde{\gamma}^\Delta) = 0$, which, along with (78), implies the validity of (77).

By (41), $\max_{\gamma \in W_N^\Delta} \rho(\gamma, W_N) = 0$. Hence, to prove (42), it is enough to establish that

$$(79) \quad \lim_{\Delta \rightarrow 0} \max_{\gamma \in W_N^\Delta} \rho(\gamma, W_N^\Delta) = 0.$$

Since (as can be easily verified using the triangle inequality),

$$\max_{\gamma \in W_N^\Delta} \rho(\gamma, W_N^\Delta) \leq \max_{\gamma \in W_N} \rho(\gamma, Z_N^\Delta) + \max_{\gamma \in Z_N^\Delta} \rho(\gamma, W_N^\Delta)$$

and since (77) has been already verified, equality (79) will be established if one shows that

$$(80) \quad \lim_{\Delta \rightarrow 0} \max_{\gamma \in Z_N^\Delta} \rho(\gamma, W_N^\Delta) = \lim_{\Delta \rightarrow 0} \rho(\bar{\gamma}^\Delta, W_N^\Delta) = 0,$$

where $\bar{\gamma}^\Delta = \{\bar{\gamma}_{l,k}^\Delta\} \in Z_N^\Delta$ is such that $\rho(\bar{\gamma}^\Delta, W_N^\Delta) = \max_{\gamma \in Z_N^\Delta} \rho(\gamma, W_N^\Delta)$ for any $\Delta > 0$.

Let $q_j(\cdot)$ be the same as in definition (15) of the metric ρ . Consider the following finite-dimensional linear program:

$$(81) \quad F_J(\Delta) \stackrel{\text{def}}{=} \min_{\gamma = \{\gamma_{l,k}\} \in W_N^\Delta} \sum_{j=1}^J \frac{1}{2^j} \left| \sum_{l,k} q_j(u_l, y_k) \gamma_{l,k} - \sum_{l,k} q_j(u_l, y_k) \bar{\gamma}_{l,k}^\Delta \right|.$$

To prove that (80) is valid, it is enough to show that

$$(82) \quad \lim_{\Delta \rightarrow 0} F_J(\Delta) = 0, \quad J = 1, 2, \dots$$

Below it is shown that the optimal value of the problem dual to (81) tends to zero as Δ tends to zero. Since the latter coincides with $F_J(\Delta)$, this will prove (82). Also, from (82) it follows that $F_J(\Delta)$ is bounded and, hence, W_N^Δ is not empty for Δ small enough (see, e.g., Theorem 2 on page 129 in [18]).

Let us rewrite problem (81) in the equivalent form:

$$(83) \quad F_J(\Delta) = \min_{\gamma = \{\gamma_{l,k}\} \in W_N^\Delta} \sum_{j=1}^J \frac{1}{2^j} \theta_j,$$

where

$$(84) \quad - \sum_{l,k} q_j(u_l, y_k) \gamma_{l,k} + \theta_j \geq - \sum_{l,k} q_j(u_l, y_k) \bar{\gamma}_{l,k}^\Delta,$$

$$(85) \quad \sum_{l,k} q_j(u_l, y_k) \gamma_{l,k} + \theta_j \geq \sum_{l,k} q_j(u_l, y_k) \bar{\gamma}_{l,k}^\Delta.$$

The problem dual to (83)–(85) is

$$(86) \quad F_J(\Delta) = \max_{\lambda_i, \mu_j, \eta_j, \zeta} \sum_{j=1}^J (-\mu_j + \eta_j) \left(\sum_{l,k} q_j(u_l, y_k) \bar{\gamma}_{l,k}^\Delta \right) + \zeta,$$

where $\lambda_i, i = 1, \dots, N$; $\mu_j, \eta_j, j = 1, \dots, J$, and ζ satisfy the following relationships:

$$(87) \quad \sum_{i=1}^N \lambda_i (\phi'_i(y^k))^T f(u_l, y_k) + \sum_{j=1}^J (-\mu_j + \eta_j) q_j(u_l, y_k) + \zeta \leq 0,$$

$l = 1, 2, \dots, L^\Delta, k = 1, 2, \dots, K^\Delta$, and

$$(88) \quad \mu_j + \eta_j = \frac{1}{2^j}, \quad \mu_j \geq 0, \eta_j \geq 0, j = 1, 2, \dots, J.$$

Before proving (82), let us verify that $F_J(\Delta)$ is bounded for Δ small enough (which, by (81), is equivalent to that W_N^Δ is not empty). Assume it is not. Then there exist a sequence $\Delta^r, r = 1, 2, \dots, \lim_{r \rightarrow \infty} \Delta^r = 0$, and sequences $\lambda_i^r, \mu_j^r, \eta_j^r, \zeta^r$, satisfying (87)–(88) with $\Delta = \Delta^r, r = 1, 2, \dots$, such that $\lim_{r \rightarrow \infty} (\zeta^r + \sum_{i=1}^N |\lambda_i^r|) = \infty$ and

$$\lim_{r \rightarrow \infty} \frac{\zeta^r}{\zeta^r + \sum_{i=1}^N |\lambda_i^r|} \stackrel{\text{def}}{=} a \geq 0, \quad \lim_{r \rightarrow \infty} \frac{\lambda_i^r}{\zeta^r + \sum_{i=1}^N |\lambda_i^r|} \stackrel{\text{def}}{=} v_i,$$

where

$$(89) \quad a + \sum_{i=1}^N |v_i| = 1.$$

Dividing (87) by $\zeta^r + \sum_{i=1}^N |\lambda_i^r|$ and passing to the limit as $r \rightarrow \infty$, one can obtain

$$(90) \quad \sum_{i=1}^N v_i (\phi'_i(y))^T f(u, y) + a \leq 0 \quad \forall (u, y) \in U \times Y,$$

where it is taken into account that every point $(u, y) \in U \times Y$ can be presented as the limit of (u_l, y_k) belonging to the sequence of cells $Q_{l,k}^{\Delta^r}$ such that $(u, y) \in Q_{l,k}^{\Delta^r}$.

Two cases are possible: $a > 0$ and $a = 0$. If $a > 0$, then the validity of (90) implies that the function $\phi(y) \stackrel{\text{def}}{=} \sum_{i=1}^N v_i \phi_i(y)$ satisfies (19) which would lead to W_N being empty. The set W_N , however, is not empty (by our assumption) and, hence, the only case to consider is $a = 0$. In this case, (90) becomes

$$(91) \quad \sum_{i=1}^N v_i (\phi'_i(y))^T f(u, y) \leq 0 \quad \forall (u, y) \in U \times Y.$$

By Assumption 2, (91) can be valid only with all v_i being equal to zero. This contradicts (89) and, thus, proves that $F_J(\Delta)$ is bounded for Δ small enough (and that W_N^Δ is not empty).

From the fact that $F_J(\Delta)$ is bounded it follows that a solution $\lambda_i^\Delta, i = 1, \dots, N$; $\mu_j^\Delta, \eta_j^\Delta, j = 1, \dots, J$, and ζ^Δ of the problem (86)–(88) exists. Using this solution,

one can obtain the following estimates:

$$\begin{aligned}
 0 \leq F_J(\Delta) &= \sum_{j=1}^J (-\mu_j^\Delta + \eta_j^\Delta) \left(\sum_{l,k} q_j(u_l, y_k) \bar{\gamma}_{l,k}^\Delta \right) + \zeta^\Delta \\
 &= \sum_{l,k} \bar{\gamma}_{l,k}^\Delta \left(\sum_{j=1}^J (-\mu_j^\Delta + \eta_j^\Delta) q_j(u_l, y_k) \right) + \zeta^\Delta \\
 &\leq \sum_{l,k} \bar{\gamma}_{l,k}^\Delta \left(- \sum_{i=1}^N \lambda_i^\Delta (\phi'_i(y^k))^T f(u_l, y_k) - \zeta^\Delta \right) + \zeta^\Delta \\
 &= - \sum_{i=1}^N \lambda_i^\Delta \left(\sum_{l,k} (\phi'_i(y^k))^T f(u_l, y_k) \bar{\gamma}_{l,k}^\Delta \right) \leq \sum_{i=1}^N |\lambda_i^\Delta| \kappa(\Delta),
 \end{aligned}$$

where the last two relationships are implied by the fact that $\bar{\gamma}^\Delta = \{\bar{\gamma}_{l,k}^\Delta\} \in Z_N^\Delta$ (see (76)).

To prove (82), it is now sufficient to show that $\sum_{i=1}^N |\lambda_i^\Delta|$ remains bounded as $\Delta \rightarrow 0$. Assume it is not. Then there exists a sequence $\Delta^r, r = 1, 2, \dots, \lim_{r \rightarrow \infty} \Delta^r = 0$, and sequences $\lambda_i^r, \mu_j^r, \eta_j^r, \zeta^r$, satisfying (87)–(88) with $\Delta = \Delta^r, r = 1, 2, \dots$, such that

(92)

$$\lim_{r \rightarrow \infty} \sum_{i=1}^N |\lambda_i^r| = \infty, \quad \lim_{r \rightarrow \infty} \frac{\zeta^r}{\sum_{i=1}^N |\lambda_i^r|} = 0, \quad \lim_{r \rightarrow \infty} \frac{\lambda_i^r}{\sum_{i=1}^N |\lambda_i^r|} \stackrel{\text{def}}{=} v_i, \quad \sum_{i=1}^N |v_i| = 1.$$

Dividing (87) by $\sum_{i=1}^N |\lambda_i^r|$ and passing to the limit as $r \rightarrow \infty$, one obtains that the inequality (91) is valid, which, by Assumption 2, implies that $v_i = 0, i = 1, \dots, N$. This contradicts the last equality in (92) and, thus, proves (82). \square

Proof of Proposition 10. Assume that (46) is not true. Then there exist a number $r > 0$ and a sequence N_i tending to infinity as i tends to infinity and there exist sequences $\Delta_{i,j}$, with each $\Delta_{i,j}$ tending to zero as j tends to infinity (with i being fixed) such that

$$\begin{aligned}
 (93) \quad \gamma_{N_i}^{\Delta_{i,j}}((U \times Y)/(\Theta + rB)) &= \gamma_{N_i}^{\Delta_{i,j}}(\Theta_{N_i}^{\Delta_{i,j}}/(\Theta + rB)) \geq \delta \\
 &\Rightarrow \gamma_{N_i}^{\Delta_{i,j}}(\Theta + rB) < 1 - \delta.
 \end{aligned}$$

Due to compactness of $\mathcal{P}(U \times Y)$, one may assume (without loss of generality) that there exists $\gamma_{N_i} \in \mathcal{P}(U \times Y)$ such that $\lim_{j \rightarrow \infty} \rho(\gamma_{N_i}^{\Delta_{i,j}}, \gamma_{N_i}) = 0$. Hence, since $\Theta + rB$ is an open set,

$$(94) \quad \liminf_{j \rightarrow \infty} \gamma_{N_i}^{\Delta_{i,j}}(\Theta + rB) \geq \gamma_{N_i}(\Theta + rB).$$

By Proposition 9, γ_{N_i} is a solution of problem (30). Consequently, by Proposition 7 and the fact that γ^* is the unique solution of (21),

(95)

$$\lim_{i \rightarrow \infty} \rho(\gamma_{N_i}, \gamma^*) = 0 \quad \Rightarrow \quad \liminf_{i \rightarrow \infty} \gamma_{N_i}(\Theta + rB) \geq \gamma^*(\Theta + rB) = 1.$$

The relationships (94) and (95) imply that, for i and j large enough, $\gamma_{N_i}^{\Delta_{i,j}}(\Theta + rB) \geq 1 - \delta$. This contradicts (93) and, thus, proves (46). The inclusion (47) follows from (46). \square

Proof of Proposition 11. Assume that the proposition is not true. Then there exist a number $r > 0$ and sequence: $(u_i, y_i) \in \Theta$, N_i , $\Delta_{i,j}$, $i = 1, 2, \dots$, $j = 1, 2, \dots$, with

$$\lim_{i \rightarrow \infty} (u_i, y_i) = (\bar{u}, \bar{y}) \in cl\Theta, \quad \lim_{i \rightarrow \infty} N_i = \infty, \quad \lim_{j \rightarrow \infty} \Delta_{i,j} = 0$$

such that

$$(96) \quad d((u_i, y_i), \Theta_{N_i}^{\Delta_{i,j}}) \geq r \quad \Rightarrow \quad d((\bar{u}, \bar{y}), \Theta_{N_i}^{\Delta_{i,j}}) \geq \frac{r}{2},$$

where $d((u, y), Q)$ stands for the distance between a point $(u, y) \in U \times Y$ and a set $Q \subset U \times Y$: $d((u, y), Q) \stackrel{\text{def}}{=} \inf_{(u', y') \in Q} \{ \|(u, y) - (u', y')\| \}$. The second inequality in (96) implies that

$$\left((\bar{u}, \bar{y}) + \frac{r}{2}B \right) \cap \Theta_{N_i}^{\Delta_{i,j}} = \emptyset \quad \Rightarrow \quad \gamma_{N_i}^{\Delta_{i,j}} \left((\bar{u}, \bar{y}) + \frac{r}{2}B \right) = 0.$$

Similarly to the proof of Proposition 10, one may assume, without loss of generality, that there exists $\gamma_{N_i} \in \mathcal{P}(U \times Y)$ such that $\lim_{j \rightarrow \infty} \rho(\gamma_{N_i}^{\Delta_{i,j}}, \gamma_{N_i}) = 0$. Hence, since the set $(\bar{u}, \bar{y}) + \frac{r}{2}B$ is open,

$$0 = \lim_{j \rightarrow \infty} \gamma_{N_i}^{\Delta_{i,j}} \left((\bar{u}, \bar{y}) + \frac{r}{2}B \right) \geq \gamma_{N_i} \left((\bar{u}, \bar{y}) + \frac{r}{2}B \right).$$

As in the proof of Proposition 10, γ_{N_i} is a solution of problem (30) (see Proposition 9). Consequently, from Proposition 7 and from the fact that γ^* is the unique solution of (21) it follows that

$$\lim_{i \rightarrow \infty} \rho(\gamma_{N_i}, \gamma^*) = 0 \quad \Rightarrow \quad 0 = \lim_{i \rightarrow \infty} \gamma_{N_i} \left((\bar{u}, \bar{y}) + \frac{r}{2}B \right) \geq \gamma^* \left((\bar{u}, \bar{y}) + \frac{r}{2}B \right).$$

The latter contradicts Assumption 3 and, thus, proves the proposition. □

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