 Remark. The theorem can be generalized as follows: for any sequence $1 < k_1 < \cdots < k_n$ there exists a polynomial

$$S_n = \sum_{s=1}^{n} \delta_{k_s},$$

with properties similar to (2) and (3).

References


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A Simple Proof of the Beloshapka Theorem on the Parametrization of the Automorphism Group of a CR Manifold

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Key words: CR manifold, Levy nondegenerate manifold, infinitesimal automorphism, nondegenerate quadric.

Let $M \subset \mathbb{C}^{n+k}$ be a real analytic CR manifold of codimension $k$ and of CR dimension $n$. In appropriate coordinates $(z^1, \ldots, z^n, w^1 = u^1 + iv^1, \ldots, w^k = u^k + iv^k)$, $M$ is described by an equation $v = F(z, \bar{z}, u)$, where $F(z, \bar{z}, u)$ is an $\mathbb{R}^k$-valued real analytic function, which can be written in the form

$$F(z, \bar{z}, u) = \sum_{i=0}^{\infty} F_i(z, \bar{z}, u),$$

where $F_i(tz, t\bar{z}, t^2 u) = t^i F_i(z, \bar{z}, u)$. By a change of variables, the terms $F_0$ and $F_1$ can be eliminated and the term $F_2$ reduced to the form $F_2 = \langle z, z \rangle$, where $\langle \cdot, \cdot \rangle$ is a $\mathbb{C}^k$-valued Hermitian form on $\mathbb{C}^n$.

The manifold $M$ is said to be *Levy nondegenerate* at the point 0 if the following conditions hold:

1) the relation $\langle a, z \rangle = 0$ for all $z \in \mathbb{C}^n$ implies $a = 0$;
2) the forms $\langle \cdot, \cdot \rangle^w$ are linearly independent.

Denote by $G$ the group of local isotropic automorphisms of $M$, i.e., of the germs of biholomorphic mappings $\varphi$ at 0 such that $\varphi(0) = 0$ and $\varphi(M) \subset M$. In 1990, Beloshapka obtained the following result [1].

Theorem 1. Let $M$ be a real analytic CR manifold nondegenerate at 0 and let $\varphi \in G$. Then $\varphi = (f, g)$ is uniquely defined by its first and second derivatives at 0.
Proof. As in the original proof, by using the Moser scheme [2], we derive a system of differential equations. Let \( \varphi(z, w) = (z^* = f(z, w), w^* = g(z, w)) \), and let

\[
\begin{align*}
f &= \sum_{q=0}^{\infty} f_q, \\
g &= \sum_{q=0}^{\infty} g_q,
\end{align*}
\]

where \( f_q(tz, t^2w) = t^q f_q(z, w) \) and \( g_q(tz, t^2w) = t^q g_q(z, w) \).

Then we have \( f_0 = 0, f_1 = Cz, g_0 = g_1 = 0, \) and \( g_2 = \rho w \), where \( C \in \text{GL}(n, \mathbb{C}) \) and \( \rho \in \text{GL}(k, \mathbb{R}) \) satisfy \( (Cz, Cz) = \rho(z, z) \).

The condition \( \varphi(M) \subset M \) means that

\[
\text{Im} g(z, w) = \langle f(z, w), f(z, w) \rangle + \sum_{l=3}^{\infty} F_l(f, \bar{f}, \text{Re} g)
\]

for

\[
\text{Im} w = (z, z) + \sum_{l=3}^{\infty} F_l(z, \bar{z}, \text{Re} w).
\]

On selecting the \( \mu \)th component, we obtain

\[
\text{Re}(i g_\mu + 2(f_{\mu-1}, Cz))|_{v=(z, z)} = F_\mu(z, \bar{z}, u) - F_\mu(Cz, \bar{Cz}, \rho u) + \cdots,
\]

where the dots stand for expressions that depend only on \( F_\nu, g_\nu, \) and \( f_{\nu-1} \) with \( \nu < \mu \). This recurrent formula uniquely determines \( g_\mu \) and \( f_{\mu-1} \) if the lower components are already known and if the corresponding homogeneous system

\[
\text{Re}(i g_\mu + 2(f_{\mu-1}, Cz))|_{v=(z, z)} = 0
\]

has the trivial solution only. After replacing \( f \) by \( Cf \) and \( g \) by \( \rho g \), this system becomes

\[
\text{Re}(i g_\mu + 2(f_{\mu-1}, z))|_{v=(z, z)} = 0. \tag{1}
\]

Collecting the components of degree \( p \) with respect to \( z \) and of degree \( q \) with respect to \( \bar{z} \) in (1) and performing algebraic manipulations, and denoting now by \( f_p \) and \( g_p \) the monomials of degree \( p \) with respect to \( z \) in the old expressions \( f_{\mu-1} \) and \( g_\mu \), we see that \( g_p = 0 \) for \( p \geq 2 \), \( f_p = 0 \) for \( p \geq 3 \), \( \text{Im} g_0 = 0 \), and

\[
\begin{align*}
g_1 &= 2i(z, f_0), \\
2 \text{Re}(f_1, z) &= \text{Re} \Delta g_0, \\
\langle f_2, z \rangle &= 2i(z, \Delta f_0), \\
\text{Im}(\Delta f_1, z) &= 0, \\
\langle z, \Delta^2 f_0 \rangle &= 0, \\
\text{Re} \Delta^3 g_0 &= 0,
\end{align*}
\]

where

\[
\Delta = \sum_{\kappa=1}^{k} (z, z)^{\kappa} \frac{\partial}{\partial u^{\kappa}}
\]

(cf. [1]).

For \( k > 1 \), a system of partial differential equations with constant coefficients with respect to \( u \) is obtained (for a chosen \( z \)). To solve this system, Beloshapka applied the Palamodov theorem on the
exponential representation of solutions to such systems [3, pp. 289–321 of the Russian edition]. However, the following simple approach makes it possible to avoid this application. It immediately follows from (7) that \( \Re g_0 \), and \( g_0 \) as well, is a polynomial of degree at most 2 with respect to \( u \). Relations (3) and (5) imply

\[
\langle \Delta^2 f_1, z \rangle = 0.
\]

Now we derive from (8) the fact that \( f_1 \) linearly depends on \( u \). Since the question is in the polynomial solutions, we may assume that \( f_1 \) is a polynomial in \( u \) and is linear in \( z \). The Fourier transform of (8) with respect to is

\[
\sum_{|m|=0}^{M} \left( \sum_{\nu=1}^{n} \alpha^\nu_m z^\nu \right) \langle (z, z), \xi \rangle^2 D^m \delta = 0,
\]

where \( \xi \) is the variable dual to \( u \), \( \delta \) is the delta function, \((\cdot, \cdot)\) is the standard inner product in \( \mathbb{R}^k \), \( m = (m_1, \ldots, m_k) \) are multiindices with \(|m| = m_1 + \cdots + m_k\),

\[
D^m = \frac{\partial^{|m|}}{(\partial u^1)^{m_1} \cdots (\partial u^k)^{m_k}},
\]

and \( \alpha^\nu_m \) are constant \( \mathbb{C}^n \)-vectors.

Without loss of generality, we may assume that \( M \) is the largest number such that there exists a nonzero \( \alpha^\nu_m \) with \(|m| = M \). Then \( M \) is equal to the degree of the polynomial \( f_1 \) with respect to \( u \). Among all nonzero \( \alpha^\nu_m \) with \(|m| = M \) we choose those for which \( m_1 \) is maximal, among the latter we choose those for which \( m_2 \) is maximal, and so on. Thus, we arrive at a nonzero matrix-valued coefficient \( \alpha^\nu_m = (\alpha^\nu_{m_1}) \), which is defined uniquely. Assume that \( M \geq 2 \). Then we apply the functional from the left-hand side of (9) to the following \( \mathbb{R}^k \)-valued test function \( \psi \). Let \( r, r \leq k \), be the maximal subscript with \( m_r \neq 0 \). For \( m_r \geq 2 \), we set \( \psi = \psi_0 \xi_{m_1}^1 \cdots \xi_{m_r}^{r-2} \).

Otherwise, we introduce \( s, s \leq r \), as the maximal subscript for which \( m_s \neq 0 \). Then we set \( \psi = \psi_0 \xi_{m_1}^1 \cdots \xi_{m_s}^{s-1} \). The vector \( \psi_0 \) is defined below.

According to the choice of \( \psi \), we have

\[
\left( \sum_{\nu=1}^{n} \alpha^\nu_m z^\nu, \zeta \right) \langle (z, \zeta)^r, \psi_0 \rangle = 0
\]

or

\[
\left( \sum_{\nu=1}^{n} \alpha^\nu_m z^\nu, \zeta \right) \langle (z, \zeta)^s, \psi_0 \rangle = 0,
\]

respectively (we have replaced the antiholomorphic \( z \)-variables by \( \zeta \)).

Let us choose \( z_0 \in \mathbb{C}^n \) so that

\[
\sum_{\nu=1}^{n} \alpha^\nu_m z_0^\nu \neq 0.
\]

Then, according to 1), there exists a \( \zeta_0 \) such that

\[
\left( \sum_{\nu=1}^{n} \alpha^\nu_m z_0^\nu, \zeta_0 \right) \neq 0.
\]

By continuity, inequality (12) holds for \( z \) and \( \zeta \) in sufficiently small neighborhoods of \( z_0 \) and \( \zeta_0 \) as well. By 2), there exist \( z_1 \) and \( \zeta_1 \) in these neighborhoods such that \( (z, \zeta)^r \) and \( (z, \zeta)^s \) do not vanish. Then for an appropriate \( \psi_0 \), the left-hand side of (11) (of (12), respectively) does not vanish. A contradiction. Therefore, the assumption \( M \geq 2 \) is wrong. Hence, the polynomial \( f_1 \) is linear in \( u \).

Exactly in the same way, from (6) we can derive the fact that \( f_0 \) linearly depends on \( u \). Taking account of (2), this means that \( g_1 \) is linear in \( u \) as well.
Let us write out the germ $\varphi$ in the form

$$f = f_0 + f_1 + f_2 + \cdots, \quad g = g_0 + g_1 + \cdots,$$

where the dots denote expressions that are determined by $f_0$, $f_1$, $f_2$, $g_0$, and $g_1$. It follows from (4) that $f_2$ is defined by $f_0$, and it follows from (2) that $g_1$ is defined by $f_0$ as well. According to (3) and (5), $f_1$ is defined by $g_0$.

Thus,

$$f = z + aw + \cdots, \quad g = w + r(w, w) + \cdots.$$  

As a consequence of the solution of system (1), Beloshapka [4] obtained an explicit description of the infinitesimal automorphisms of the nondegenerate quadrics $Q = \{(z, w) : \text{Im} w = \langle z, z \rangle\}$. These infinitesimal automorphisms turn out to be holomorphic vector fields

$$\chi = \sum_{\nu=1}^{n} \zeta^{\nu} \frac{\partial}{\partial z^{\nu}} + \sum_{\kappa=1}^{k} \omega^{\kappa} \frac{\partial}{\partial w^{\kappa}},$$

such that

$$\text{Re} \chi(v - (z, z))|_{v = (z, z)} = 0.$$

The coefficients $\zeta = (\zeta^{\nu})$ and $\omega = (\omega^{\kappa})$ satisfy system (1), and therefore, these are quadratic functions $\zeta = \zeta_0 + \z_1 + \z_2$ and $\omega = \omega_0 + \omega_1$, where

$$\z_0 = p + aw, \quad \z_1 = Xz + B(w, z), \quad \z_2 = A(z, z), \quad \omega_0 = q + sw + r(w, w), \quad \omega_1 = 2i\langle z, p \rangle + 2i\langle z, aw \rangle,$$

$p \in \mathbb{C}^n$, $q \in \mathbb{R}^k$, the pairs $(X, s) \in \text{GL}(n, \mathbb{C}) \times \text{GL}(k, \mathbb{R})$ satisfy the condition $2\text{Re}\langle Xz, z \rangle = s\langle z, z \rangle$ for $z \in \mathbb{C}^n$, $a$ is a linear mapping from $\mathbb{C}^k$ into $\mathbb{C}^n$, $A(z, z)$ is a $\mathbb{C}^n$-valued symmetric bilinear form on $\mathbb{C}^n$ such that

$$\langle A(z, z), z \rangle = 2i\langle z, a(z, z) \rangle \quad \text{for} \quad z \in \mathbb{C}^n,$$

$B(w, z)$ is a $\mathbb{C}^n$-valued bilinear form on $\mathbb{C}^k \otimes \mathbb{C}^n$, and $r(w, w)$ is an $\mathbb{R}^k$-valued symmetric bilinear form on $\mathbb{R}^k$ such that

$$\text{Re}\langle B(u, z), z \rangle = r(u, \langle z, z \rangle), \quad \text{Im}\langle B((z, z), z), z \rangle = 0 \quad \text{for} \quad z \in \mathbb{C}^n, \quad u \in \mathbb{R}^k.$$  

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